Moving the Goalposts*

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Abstract

We study information as an incentive device in a dynamic moral hazard framework. An agent works on a task of uncertain difficulty, modeled as the duration of required effort. The principal knows the task difficulty and can provide information over time with the goal of inducing maximal effort. The optimal mechanism features moving goalposts: an initial disclosure makes the agent sufficiently optimistic that the task is easy in order to induce him to start working. If the task is indeed difficult the agent is told this only after working long enough to put the difficult task within reach. Then the agent completes the difficult task even though he never would have chosen to at the outset. The value of dynamic disclosure implies that principal prefers a random threshold over any deterministic scheme. We consider extensions to two-player pre-emption games and bandits. Keywords: goalposts, leading him on, information design.

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1 Introduction

Monetary and other material rewards are standard instruments in the design of incentives. But when material rewards are either unavailable or have already been exhausted, there still remains a powerful motivational tool: information. In an organization where salaries and bonuses are set by higher-ups, a mid-level manager incentivizes his team by providing performance feedback and guiding productive activity. An athlete whose long-term motivation comes from prizes offered by the wider market nevertheless finds day-to-day motivation from a coach who offers nothing but information. And of course information is the exclusive currency of intra-personal incentives wherein a decision-maker might manage his own access to information in order to better align his impulses with longer-term goals.

We analyze a principal-agent problem in which the agent is faced with a task of uncertain difficulty. Difficult tasks require a longer duration of effort to complete. If the agent completes the task he earns a reward that is exogenously given and outside the control of the principal. The principal is informed about the difficulty of the task and she selectively reveals information over time to the agent about how far he is from completion. The principal’s objective is to induce the agent to work as much as possible whereas the agent trades off the reward from completing the task and the cost of effort.

The principal can commit to an arbitrary dynamic policy of information disclosure. The agent knows the policy, observes the realized disclosures, rationally updates his belief about the difficulty of the task, and optimally responds with effort choices to maximize his expected payoff from the reward net effort costs. We fully characterize the optimal policy.

The nature and timing of optimal disclosures vary across the lifetime of the project. In the late stages, when the agent has already made substantial progress, his goal is within reach even if the task is difficult. At this stage the principal wants to persuade the agent that the task is difficult so that the agent will continue working longer. However, in the early stages of the project, the difficult task may be out of reach: if the agent knew the task were difficult he would prefer to quit immediately rather than spend the effort to complete it. Thus, at the early stages of the project the principal wants to persuade the agent that the task is easy.

This reversal of incentives over the duration of the project gives rise to some novel features of optimal persuasion in our dynamic setting. When the task is not too difficult, the optimal contract involves leading the agent on. Early on, the agent would
like to know whether the task is difficult because in that case he would rather quit immediately. Instead the principal is purposefully silent and reveals nothing about the difficulty of the task. If indeed the task is difficult the principal waits until the agent has worked long enough to complete the easy task and only then does she inform the agent that there is more work to do. Having reached that point, since the agent has already made substantial progress he completes the difficult task even though he would never have set out to do that in the first place.

This feature of the optimal mechanism utilizes a role for information as an incentive device that is novel and unique to dynamic persuasion: information as a carrot. When at the outset the agent is pessimistic that the task is difficult, he is unwilling to even begin working. One way to induce the agent to spend effort would be to disclose information right away to persuade the agent that the task is less difficult than he fears. Instead, the principal withholds this valuable information and only promises to reveal it as a reward for some minimal initial effort. Thus, delayed disclosure confers a double incentive effect: first, the carrot incentivizes the agent to commence working at the outset and later, the eventual disclosure induces him to continue working when the task turns out to be in fact difficult.

However, when the agent is very pessimistic, even promising the maximum value, i.e. full disclosure, is not enough to incentivize the agent to start working. In this case the optimal mechanism involves moving the goalposts. The principal begins with an initial partial disclosure that leaves the agent relatively optimistic that the task is easy and asks the agent to begin working. Then, after the agent has made substantial progress, if the task is in fact difficult the principal provides a second disclosure which informs the agent of this and induces him to work even more.

Our work sheds new light on optimal information design by demonstrating how dynamic disclosures can play a role even in a setting which, from the point of view of fundamentals, is essentially static in nature. In our model the state, i.e. the total amount of effort required to complete the task, is determined at the outset and never changes. The other dynamic state variables, effort and progress, are symmetrically observed by principal and agent. Thus, all of the information the principal ever provides could feasibly be provided in a single disclosure at the beginning of the relationship.\(^1\) Indeed there is a natural static reduced-form of our problem in which that is the only oppor-

\(^1\text{Thus, the role for dynamic disclosures is inherently different from Ely (2017). In that paper disclosure policies are necessarily dynamic because the payoff-relevant variables privately observed by the principal are themselves changing over time.}\)
portunity to provide information, after which the agent chooses how much effort to spend and the game ends. We show how the optimal dynamic contract strictly improves on the best the principal could achieve in that static reduced-form.

Next we endogenize the task difficulty. We consider a principal who can choose the prior distribution over tasks and then designs an information policy. In the context of a junior partner who is working toward a promotion we are asking whether the promotion standard should be transparent and known at the outset or vague and clarified only partially and gradually over time. Task uncertainty, by unleashing the benefits of dynamic information disclosure can improve over the optimal deterministic mechanism, especially when the principal is patient.

We pursue a number of further extensions. We consider an alternative formulation in which the task is known but the agent’s productivity is uncertain and we consider the case in which the principal shares in the reward. The analysis carries through with minor changes. We consider a setting of random breakthroughs. Here the task is completed only when the agent achieves a breakthrough and neither principal or agent know in advance how much effort this requires. The principal observes the arrival rate of breakthroughs and decides when to reveal it to the agent. Finally we consider dynamic persuasion in a stylized promotion tournament between two agents.

1.1 Related Literature

We contribute to the literature on information design and Bayesian persuasion, see Bergemann and Morris (2016b), Bergemann and Morris (2016a), Kamenica and Gentzkow (2011), and Aumann, Maschler and Stearns (1995). We apply these tools to a dynamic setting with moral hazard. A closely related paper is Smolin (2015) who studies optimal performance feedback in a Principal/Agent framework. The Principal designs feedback to persuade the agent that he is high ability in order maintain the agent’s incentives to keep working. Ability is persistent in Smolin (2015) and so the incentive reversals that drive the key qualitative features of our model do not arise. See also Hansen (2012), Fang and Moscarini (2005), Horner and Lambert (2016), Orlov, Skrzypacz and Zryumov (2016), and Varas, Marinovic and Skrzypacz (2017).

Halac, Kartik and Liu (2016) study the design of disclosure policies in a dynamic contest with multiple agents. See also Lizzeri, Meyer and Persico (1999), Aoyagi (2010) and Ederer (2010). While these papers compare some specific disclosure policies, a goal of future research would be to apply information design methods to characterize opti-
mal disclosure policies in contests. Our application to pre-emption games in Section 6.3 makes a small step in this direction.

Our work can also be seen as a contribution to optimal incentive contracts without transfers. Ben-Porath, Dekel and Lipman (2014), Bull and Watson (2007) and Ben-Porath, Dekel and Lipman (2017) examine disclosure policies as incentive schemes in a setting of verifiable information. For a broad survey, see Dekel (2017).

2 Model

An agent works on behalf of a principal and spends effort continuously until he chooses to quit. At that point the relationship ends and the agent earns a reward $R > 0$ if and only if his total accumulated output exceeds a threshold $x > 0$. We refer to the threshold as the difficulty of the task. We normalize the production technology to accrue one unit of output per instant of effort at flow cost $r \cdot c$ to the agent.\footnote{Normalizing the flow cost by $r$ helps simplify the notation throughout the paper. We do the same for the principal’s flow value.} The realized threshold is unknown to the agent at the outset and the agent’s prior is given by the CDF $F$. Thus, the total (ex ante) expected payoff to the agent from working until time $\tau > 0$ is

$$F(\tau) e^{-r\tau} R - c(1 - e^{-r\tau}).$$

The principal is the residual claimant of the agent’s output until the time the agent quits. We normalize the principal’s flow value of output to $r$ so that the principal earns total discounted payoff $(1 - e^{-r\tau})$ when the agent works until $\tau$.

The principal observes the realized threshold $x$ at the beginning of the relationship and chooses how and when to disclose information to the agent about the difficulty of the task. The principal can make any number of arbitrary disclosures at any time in the process and we refer to the rule governing these disclosures as the information policy. Crucially we assume that the principal commits to an information policy and that the agent knows the policy and understands the principal’s commitment. Our goal is to characterize the optimal policy.

The optimal policy is designed to induce the agent to work as long as possible (in expectation) in order to maximize the payoff of the principal. The time structure of disclosures engages the following tradeoff. Informing the agent that the task is easy incentivizes the agent to continue working. However, a policy of announcing that the
task is easy entails the downside of making the agent pessimistic in the absence of the announcement, possibly inducing the agent to quit. The optimal policy balances these gains and losses at every instant while the relationship is ongoing.

We begin with a simple example to illustrate the main ideas.

3 Binary Example

In this example the task is either easy, with threshold $x_l > 0$ or hard, with threshold $x_h > x_l$. The principal knows the task difficulty. The agent is uncertain and begins with prior probability $\mu$ that the task is easy.

The hard task is not individually rational for the agent. In particular, the maximum individually rational effort $\bar{\tau}$ is given by

$$e^{-r\bar{\tau}}R - c (1 - e^{-r\bar{\tau}}) = 0,$$

and $x_l < \bar{\tau} < x_h$. Absent any intervention by the principal, the agent would never work longer than $x_l$. Nevertheless the hard task is not too hard: the incremental effort is individually rational: $x_h - x_l \leq \bar{\tau}$.

**Static Disclosure** Consider as a benchmark the static reduced-form of the problem in which there is a single disclosure by the principal after which the agent chooses effort. Because $x_h > \bar{\tau}$, the agent would not set out to complete the hard task regardless of the his belief $\mu$. Therefore there is no one-shot disclosure policy that could persuade the agent to work until $x_h$. The best the principal can do with such a policy is to persuade the agent that the task is easy. Indeed, if the agent is led to believe that with at least probability

$$\tilde{\mu} = \frac{c}{R} \left( \frac{1 - e^{-rx_l}}{e^{-rx_l}} \right)$$

the task is easy, he will optimally choose effort $\tau = x_l$. Any more pessimistic belief will induce the agent to quit immediately and earn zero. The optimal one-shot, (or static) disclosure policy is designed to minimize the probability that the agent is as pessimistic as that.

**Proposition 1.** No static policy can induce the agent to complete the hard task. The optimal static policy is the one that maximizes the probability that the agent has belief at least $\tilde{\mu}$ and spends effort $\tau = x_l$. 

This static version of the problem is equivalent to a standard Bayesian Persuasion problem as studied by Kamenica and Gentzkow (2011) and Aumann, Maschler and Stearns (1995). The principal optimally sends two messages: an optimistic message which persuades the agent to believe that with exactly probability \( \bar{\mu} \) the task is easy and a pessimistic message which convinces the agent the task is hard. The optimistic message is sent whenever the task is truly easy and also with positive probability conditional on the task being hard. The latter probability represents the principal’s gain from persuasion. This probability is maximized subject to the constraint that the resulting pooled message leaves the agent with a belief at least \( \bar{\mu} \), and this constraint binds.

**Dynamic Disclosure**  Dynamic disclosure policies can do better for a couple of reasons. First, information is valuable to the agent and that value is often best withheld and offered only later as a carrot to incentivize early effort. Second, having reached that later date the cost of past effort is sunk and the principal may then find it possible, and indeed in her interest to persuade the agent to continue working. Based on these ideas we will show how to structure dynamic disclosure to induce the agent to complete the hard task and in fact to complete the task with probability 1 regardless of its difficulty.

Suppose the principal were to delay all disclosure until time \( x_l \) and then to fully disclose the threshold. The agent would quit upon hearing that the task was easy (and already complete) and continue working until \( x_h \) upon hearing the task was difficult (because \( x_h - x_l \leq \bar{\tau} \)). Of course that raises the question of how the principal could have induced the agent to reach \( x_l \) in the first place. Indeed as the static analysis revealed, when the agent begins with a prior below \( \bar{\mu} \) he would quit immediately unless persuaded that the task were easy. With dynamic disclosure however, information as a carrot can substitute for information as persuasion. Consider the time-zero value to the agent of full disclosure at time \( x_l \):

\[
V(\mu) = e^{-rx_l} \left\{ \mu R + (1 - \mu) \left[ R e^{-(x_h - x_l)} - c \left( 1 - e^{-(x_h - x_l)} \right) \right] \right\}.
\]

With probability \( \mu \) the agent learns that the task is already complete and therefore quits and earns the reward. With probability \( 1 - \mu \) the agent learns that the task is hard. The incremental effort necessary to complete the difficult task is individually rational and the agent optimally does so, incurring additional effort costs \( c \left( 1 - e^{-(x_h - x_l)} \right) \) and earning the reward at date \( x_h \).
Leading the agent on  For all priors exceeding $\hat{\mu}$ defined by $V(\hat{\mu}) = c \cdot (1 - e^{-rx_l})$, information provided at date $x_l$ is of sufficient value to incentivize the agent to begin working.\footnote{Note that $\hat{\mu}$ is the smallest belief at which the agent would begin working without any incentive, hence $\hat{\mu} < \tilde{\mu}$.} No initial persuasion is necessary. Indeed full delayed disclosure eventually induces the agent to complete the task with probability 1. This includes the difficult task which, had he known at the outset, he would never have set out to complete.

We refer to this device of using the value of information as a carrot and delaying persuasion until effort costs are sunk as leading the agent on. Within this class of policies, the extreme of full delayed disclosure may be more incentive than necessary, i.e. when $\mu > \hat{\mu}$. In that case further fine-tuning of this mechanism utilizes persuasion at time $x_l$ to increase the probability that the agent works through $x_h$. This is illustrated in Figure 1.

\begin{figure}[h]
\centering
\begin{subfigure}{.45\textwidth}
\centering
\includegraphics[width=\textwidth]{agent_value.png}
\caption{Agent}
\end{subfigure} \hspace{1cm}
\begin{subfigure}{.45\textwidth}
\centering
\includegraphics[width=\textwidth]{principal_value.png}
\caption{Principal}
\end{subfigure}
\caption{Extracting value}
\end{figure}

The left panel shows the value of information at time $x_l$ for the agent. The two segments of the solid upper envelope shows the agent’s continuation value function: the payoff from quitting at $x_l$ when his prior is high, and continuing to $x_h$ when his prior is low. Consider a line connecting any two points on the graph of the agent’s value function and a delayed disclosure policy involving two messages which induce the corresponding beliefs. By the law of total probability, that line gives for any prior belief the expected value of such a disclosure.

For example, the dashed green line is the value function for full disclosure. By contrast, a policy such as the one represented by the solid green line is less informative and less valuable to the agent. In particular, it induces effort $x_h$ with a less-informed
belief. This is achieved by sending the associated message not just when the task is difficult but also sometimes when the task is easy.

The solid green line in the right panel shows the value of the same policy to the principal. Compared to the full-disclosure policy, the alternative policy induces the agent to work until \( x_h \) with a larger probability and is thus better for the principal. We see that distortions of this form reduce the agent’s payoffs while raising the principal’s. The optimal delayed-disclosure mechanism will thus involve just enough distortion to reduce the agent’s ex ante value to zero.

**Proposition 2.** When the prior exceeds \( \tilde{\mu} \), leading the agent on induces him to complete the task with probability 1. The optimal such mechanism maximizes the probability that the agent works through time \( x_h \) subject to the constraint that the agent’s ex ante payoff is at least zero. Indeed this is the optimal mechanism among all dynamic disclosure policies.

The last claim in the proposition is that delayed disclosure is optimal within the entire feasible set of dynamic disclosure policies, including those that involve multiple history-contingent disclosures at staggered times, possibly inducing quitting times other than \( x_l \) or \( x_h \). This follows from two observations. First, as discussed above it holds the agent to his reservation value of zero so that no further reduction in the agent’s payoff would be implementable. Second, the policy is efficient: there is no alternative feasible effort plan that can further raise the principal’s payoff without lowering that of the agent. As we show in Section 4, these are general properties of optimal dynamic disclosure policies. The proof of efficiency for this binary example is in the Appendix.

**Moving the goalposts** Dynamic disclosure enables the principal to increase his payoff relative to static disclosure. Figure 2 plots (in blue) the value to the principal as a function of the agent’s prior from leading the agent on.

When the agent is initially so pessimistic that \( \mu < \tilde{\mu} \), even full delayed disclosure is insufficient incentive to induce the agent to start working. Thus the principal’s payoff drops to zero for such priors. We can now use ideas from Bayesian persuasion to build upon delayed disclosure and construct a new mechanism which improves for this range of priors. The green segment represents the concavification of the principal’s value function. Following Kamenica and Gentzkow (2011) and Aumann, Maschler and
Stearns (1995), the concavification shows the maximum value the principal can obtain by an initial disclosure. For priors to the left of \( \tilde{\mu} \), this initial disclosure reveals one of two possibilities. The first is that the task is certainly difficult, leading the agent to quit and yielding zero for the principal. The alternative is that the task is easy with probability \( \tilde{\mu} \), making the agent just optimistic enough begin working in anticipation of a second disclosure at time \( x_l \).

Notice the pattern of disclosures in this mechanism. The goal of the initial disclosure is to persuade that the task is easy so the agent sets out working while the goal of the second disclosure is to persuade that the task is hard so that he keeps going. We refer to this mechanism as \textit{moving the goalposts}.

\textbf{Proposition 3.} There is an optimal mechanism involving disclosure at no more than two dates. At date zero the disclosure is designed to maximize the probability that the agent begins working and at date \( x_l \) the disclosure is designed to maximize the probability that the agent continues to \( x_h \).

\textbf{The general case} The mechanisms derived above made use of the fact that the incremental effort \( x_h - x_l \) was individually rational. Thus, after the sunk effort costs of reaching \( x_l \) it was possible for the principal to persuade the agent to keep working. This would not be possible if the hard task was so hard that the incremental effort exceeded \( \tilde{r} \). However, in that case, consider a third possible threshold in between \( x_h \) and \( x_l \) such that each increment was individually rational. Indeed consider adding a fourth threshold even higher than \( x_h \) with again an individually rational increment. In these extensions, it is possible to induce the agent to complete each successive increment through disclosure policies that combine the ideas of leading the agent on and moving.
the goalposts.

In the analysis of the general model below we characterize the optimal disclosure policies for general distributions of thresholds. In addition we consider the question of how the principal would optimally design the threshold distribution. Indeed not only can the principal use information design to manage incentives given the agent’s initial incomplete information about the task, but in fact the principal benefits from that incomplete information, and strictly so when the principal is more patient than the agent, i.e. when her discount rate $r_p$ is smaller than $r$. We show in the next example how a random threshold, coupled with optimal information design, allows the principal to extract more effort from the agent than would be possible when the threshold was deterministic and known.

3.1 The Optimality of A Random Threshold

We wish to show that the random threshold, taken for granted in the preceding sections, is in fact part of an optimal mechanism when the threshold is a choice of the principal. To that end, let us begin by considering the optimal deterministic mechanism. In a deterministic mechanism the threshold $x$ is known to the agent and there is hence no role for information disclosure. The agent works if and only if $x$ is individually rational and therefore the maximal individually rational effort duration $\bar{t}$ is the optimal deterministic threshold for the principal.

We will now show how adding randomness and thereby enabling delayed disclosure expands the feasible set of implementable efforts and raises the payoff of the principal. Consider a random threshold with two possible realizations, $x_l$ and $x_h$ in which

$$
x_l = \bar{t} - \tilde{t}$$
$$x_h = \bar{t} + \tilde{t},$$

where $\tilde{t}$ is any positive duration satisfying

$$R/2 \leq e^{-r\tilde{t}} R - c(1 - e^{-r\tilde{t}}).$$

(1)

This new distribution is a mean-preserving spread of $\bar{t}$ with the gap (equal to $2\tilde{t}$) between high and low thresholds small enough that the agent would prefer to continue working from $x_l$ to $x_h$ in order to earn the reward with probability 1 rather than quit at
and earn the reward with probability 1/2. Note that 2\(\bar{\tau}\) < \(\bar{\tau}\).

Without any accompanying dynamic disclosure policy, this random threshold would be strictly worse for both parties. Intuitively, the agent was just willing to work until \(\bar{\tau}\) and earn the reward for sure. The easy task is not much easier than that but would now only yield the reward with probability 1/2. Faced with that option or the option of the non-individually-rational difficult task the agent would strictly prefer to quit immediately.

However, full delayed disclosure would induce the agent to complete the task with probability 1. The result is a lottery: with equal probability the agent works until either \(x_l\) or \(x_h\) whereupon he earns the reward. Due to discounting, the agent is risk-loving with respect to this lottery. To see this, re-arrange the agent’s utility from earning the reward at some time \(t\) as follows:

\[(c + R) e^{-rt} - c.\]

Ignoring the constant \(-c\), the agent evaluates lotteries using a strictly convex exponential utility function. He thus strictly prefers the random threshold.

The principal on the other hand is risk-averse with respect to the lottery for exactly the same reason: up to a constant his utility is \(-e^{-r_p t}\). Thus, merely adding randomness is no more than a transfer of surplus from the principal to the agent.

There is however a second channel for transferring surplus from the agent back to the principal: increase the mean of the agent’s effort. This is achieved by partial delayed disclosure. Because the random mechanism is strictly individually rational (we showed it gives the agent strictly higher payoff than the just-individually-rational deterministic threshold), the principal can use persuasion at date \(x_l\) to increase the probability the agent works until \(x_h\). Indeed, consider the partial disclosure policy which increases the probability of working until \(x_h\) just enough to return the agent’s ex ante utility to zero (c.f. Figure 1). We show in the Proposition below that the net effect of randomization plus delayed disclosure raises the expected payoff of the principal, and strictly so when \(r_p < r\).

**Proposition 4.** There exists a random threshold which, when coupled with an optimal disclosure policy, improves on the optimal deterministic threshold, and strictly so when \(r_p < r\).

For intuition, consider Figure 3 where the expected payoffs of the agent \((V)\) and principal \((W)\) are plotted on the horizontal and vertical axes. Point A represents the
optimal deterministic mechanism. The movement to point $B$ represents the increase in the agent’s payoff, call it $\Delta_1^a$, and decrease in the principal’s payoff, call it $\Delta_1^p$, when the threshold is randomized. The movement to point $C$ represents the loss to the agent $\Delta_2^a$ and gain to the principal $\Delta_2^p$, from partial delayed disclosure. The proposition states that the line from $B$ to $C$ is steeper than the line from $A$ to $B$.

For starters, consider the extreme case $r_p \to 0$ of no discounting for the principal, i.e. linear utility equal to the expected undiscounted effort duration. In this case the proposition holds because the Principal is risk-neutral with respect to the randomization and gains from persuasion; graphically the line $AB$ is horizontal. More generally, note that exponential utility is characterized by constant absolute risk affinity/aversion with the discount rates $r$ or $r_p$ as the risk coefficient. The slope comparison reduces to the comparison of $\Delta_1^p / \Delta_1^a$ with $\Delta_2^p / \Delta_2^a$ or equivalently $\Delta_1^1 / \Delta_2^1$ with $\Delta_1^2 / \Delta_2^2$. The latter are the ratios of the attitude toward the introduction of risk (i.e. the second derivative of utility) to the attitude toward a shift in “wealth” (the first derivative.) The comparison is thus determined by the comparison of the two parties’ coefficients of absolute risk aversion, i.e. the discount rate.

Given that the principal prefers a random threshold the question of the optimal distribution over thresholds naturally arises. In Section 5 we show that an optimal threshold distribution exists, we characterize it, and show that it is independent of the principal’s discount rate.

Figure 3: The value of randomness.
4 General Results

This section presents a general characterization of the optimal dynamic information design for continuous task distributions $F$. We assume that $F$ has a continuously differentiable density, denoted $f$, with full support on $[0, \infty)$, and we use $H$ to denote the hazard rate. We make two assumptions on $F$ that facilitate a clean characterization of the optimal policy:

$$\frac{f'(t)}{f(t)} < r \quad \text{(A1)}$$

and

$$H'(t) \geq 0. \quad \text{(A2)}$$

Assumption (A1) is relevant when the agent receives no information. It ensures that his marginal value of working decreases as he works longer, so that he stops the first time this value reaches zero.

Assumption (A2) captures the main intuition of our introductory example in Section 3. There, if the agent reaches the low threshold but learns that the true threshold is high, he will continue working. In fact, as time passes, he only becomes more willing to work because the high threshold moves closer. Assumption (A2), which says that the hazard rate is increasing in $t$, implies the same. If at a positive time the agent learns that he has not reached the threshold, he is more willing to continue than before.

Effort Schedules The main tool of our analysis will be an effort schedule, defined as a joint probability of effort levels and thresholds. An effort schedule is a reduced form summary of an information policy and the agent’s optimal effort response. It describes in probabilistic terms how long the principal induces the agent to work conditional on each possible realization of the threshold. In particular we will consider a probability measure $G$ over $\mathbb{R}^2$ and interpret it as describing the joint probability of the (exogenous) threshold $x$ and the (endogenous) duration of effort $\tau$. The marginal of $G$ on $x$ coincides with $F$, and for any fixed threshold, the conditional distribution $G(\cdot \mid x)$ describes the CDF of the agent’s effort duration conditional on $x$ being the true difficulty of the task.

The values to the agent and principal of an effort schedule $G$ are simply the expecta-
tions of their corresponding ex-post payoffs with respect to $G$. The agent earns ex-post payoff

\[ v(\tau, x) = \begin{cases} 
-c(1 - e^{-\tau T}), & \text{if } \tau < x \\
 e^{-\tau T} R - c \left(1 - e^{-\tau T}\right) & \text{if } \tau \geq x
\end{cases} \]

when he works for a duration $\tau$ and the realized threshold is $x$. Thus, the schedule $G$ provides the agent with expected payoff

\[ V(G) = E_G v. \]

The principal's payoff depends only on effort and is given by $w(\tau, x) = 1 - e^{-\tau T}$, so that the expected value to the principal from a schedule $G$ is

\[ W(G) = E_G w. \]

Most of the relevant effort schedules will be pure, i.e. $G(\cdot \mid x)$ assigns probability one to a single effort duration. In the case of pure effort schedules we will use the notation $g(x)$ to refer to the effort level chosen when the threshold is $x$. With slight abuse of notation, we denote the expected values under a pure schedule as $E_g v$ and $E_g w$.

While any information policy and its induced behavioral response implement a specific effort schedule, not every effort schedule is implementable by an information policy. Information is a low-powered incentive instrument. The agent is never prevented from simply ignoring all information provided by the principal and making an uninformed optimal effort choice. Thus, a necessary condition for an effort schedule to be implementable is that it provides no less than what we refer to as the agent's no-information value.

**No Information Benchmark** Consider the decision problem of the agent if the principal were not present or if she provided no information, or if the agent simply ignored any information provided. An optimal strategy would consist of simply ignoring all information provided by the principal and making an uninformed optimal effort choice. Thus, a necessary condition for an effort schedule to be implementable is that it provides no less than what we refer to as the agent's no-information value.

The effort schedule associated with the no-information benchmark is the constant
pure schedule $g(x) = \tau$ for all $x$. Denote by $V_{ni}$ the associated expected payoff for the agent. This is the agent’s no-information value.

**Full-Information Benchmark** At the opposite extreme we may consider the policy in which the agent learns the realized threshold at the outset and chooses his effort accordingly. It is represented by the pure effort schedule $\bar{g}$:

$$
\bar{g}(x) = \begin{cases} 
  x, & \text{if } e^{-rx}R - c(1 - e^{-rx}) \geq 0 \\
  0, & \text{otherwise.}
\end{cases}
$$

The agent spends just enough effort to complete the task whenever it is individually rational to do so, and otherwise he quits immediately. Note that there exists a maximum individually rational effort level which we denote by $\bar{\tau}$. It is the most difficult task that the agent would knowingly set out to complete and it is given by the equation

$$
e^{-r\bar{\tau}}R - c(1 - e^{-r\bar{\tau}}) = 0.$$

**Efficiency** A schedule $G$ is (ex ante, Pareto) efficient if there is no other schedule $G'$ such that $W(G') > W(G)$ and $V(G') \geq V(G)$.

The full-information schedule is efficient because any schedule that differs with positive probability must give the agent a strictly lower expected payoff. Indeed the full-information policy maximizes the agent’s expected payoff.

The no-information schedule, on the other hand, is typically not efficient. This can be seen from the binary example in Section 3. When the agent was pessimistic he quit immediately and earned zero. We constructed an implementable schedule which provided the same utility to the agent but strictly higher payoff to the principal.

**Individually Rational and Implementable Schedules** Individual rationality, the condition that $V(G) \geq V_{ni}$, is a necessary condition for a schedule $G$ to be implementable through an information policy. Going one step further, we may consider the incentives of the agent at any point in the course of an information policy. For each date $t$, there is an associated continuation schedule $G_t = G(\cdot \mid \tau > t)$. This is the conditional distribution of effort durations given that the agent does not quit at or prior to $t$. If $G$ is

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Note that we do not require the dominating schedule $G'$ to be implementable by an information policy.
implementable then the associated information policy must induce the agent to obey \( G \) conditional on reaching date \( t \). Since the agent could ignore any further information from the principal, a necessary condition is that doing so would be no better than the continuation value promised by \( G \):

\[
V_t(G) := E_{G_t}(\tau - t, x - t).
\]

In fact, these conditions are also sufficient.

**Lemma 1.** *Given a schedule \( G \) and a date \( t \), let \( G_t \) be the continuation schedule. Let \( V_{ni,t} \) denote the optimal continuation value for the agent in the absence of any additional information from the principal. If for every date \( t \geq 0 \),

\[
V_t(G) \geq V_{ni,t},
\]

then \( G \) is implementable.*

The logic behind **Lemma 1** is simple. The principal is using information to incentivize the agent. The principal can threaten to withhold all future planned information disclosures, leaving the agent to his own devices. If this is enough to dissuade the agent, then the principal can implement \( G \).

### 4.1 The Schedule \( g^\infty \)

Even though the threshold distribution has unbounded support, it is often possible to persuade the agent to continue working arbitrarily long and complete the task with probability 1. This is represented by the pure schedule \( g^\infty \) defined by \( g^\infty(x) = x \). Falling short of that, the principal may try to implement the schedule \( g^t \):

\[
g^t(x) = \begin{cases} 
x, & \text{if } x \leq t \\
0, & \text{otherwise.}
\end{cases}
\]

According to this schedule the agent quits immediately when the task is too difficult \( (x > t) \) but otherwise works just long enough to complete the task.

These schedules are the main elements of our analysis. They have the convenient property that, under Assumption (A2), individual rationality is not only necessary but sufficient for implementability.
Lemma 2. For $t \in [0, \infty]$, the schedule $g^t$ is implementable if and only if it is individually rational.

To understand the lemma, take the schedule $g^\infty$. Fix some positive time $s$ and suppose that without information from $s$ onward, the agent works for an additional time $\tau^0$. The agent’s continuation value under $g^\infty$ at time $s$ consists of two parts: his value until $\tau^0$ and his continuation value from working past $\tau^0$. The first part is larger than his no-information value, because he receives the reward with the same ex-ante probability, but stops working earlier on average. Thus, $g^\infty$ is implementable at $s$ if the agent’s continuation value at $s + \tau^0$ is positive. Under Assumption (A2) we can prove that the continuation value is non-decreasing over time. Therefore if the schedule was individually rational at time zero, then its continuation value is non-negative at all future dates.

The principal prefers $g^\infty$ to any finite $g^t$, since it extracts strictly more effort from the agent. However, for exactly that reason $g^\infty$ may not be implementable. Hence the analysis divides into two cases depending on whether $g^\infty$ is implementable.

4.2 Leading The Agent On

When $g^\infty$ is implementable the principal can persuade the agent to work arbitrarily long to complete the task. Still, the schedule may provide the agent with some surplus, i.e. $V(g^\infty) > V_{ni}$, in which case it is possible for the principal to improve further.

Consider a different schedule in which the agent works with probability 1 for an initial duration $t$, after which he follows some implementable continuation schedule $G$. The total value to the agent is

$$e^{-rt}V(G) - c \left(1 - e^{-rt}\right)$$

which decreases from $V(G)$ to $-c$ as $t$ increases. By choosing $t$ we can design a schedule whose total value is exactly $V_{ni}$. As we show below this schedule is implementable and, when choosing $G$ to be an initial disclosure followed by $g^\infty$, also optimal.

Proposition 5. When $g^\infty$ is implementable, there exists a $t^*$ such that the following pure sched-

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6The intuition for finite $t$ is similar.

7This probability is $\frac{F(s^0) - F(s)}{1 - F(s)}$. 

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ule is implementable and optimal for the principal.

\[ g^*(x) = \begin{cases} t^*, & \text{if } x \leq t^* \\ x, & \text{otherwise.} \end{cases} \]

In particular the agent works until at least \( t^* \) which is (weakly) larger than the no-information effort level.

The policy that implements \( g^* \) is leading the agent on. The principal tells the agent nothing unless he first works for the duration \( t^* \). At time \( t^* \) if the agent has worked throughout then the principal disclose whether the task has been completed (i.e. if \( x \leq t^* \)). If so the agent immediately quits. If not the principal proceeds to implement the continuation schedule \( g^\infty \). At each subsequent point in time the agent is told only whether or not he has just reached the threshold, providing no information about any remaining effort required if not.

Here is why the policy is implementable. First of all, the schedule \( g^\infty \) reduces to the continuation schedule \( g^\infty \begin{cases} t^*, & \text{if } x \leq t^* \\ x, & \text{otherwise.} \end{cases} \) beginning at time \( t^* \), and as we have already shown in Lemma 2, if \( g^\infty \) is individually rational it is also implementable.

All that remains is to show that the policy is implementable prior to time \( t^* \). By Lemma 1 this amounts to showing that its continuation value \( V_t(G) \) exceeds the no-information continuation value \( V_{ni,t} \) at each such time. By choosing \( t^* \) so that the expression in Equation 2 is equal to \( V_{ni,t} \), this is satisfied for \( t = 0 \). The main step of the proof involves showing that \( V_t(G) \) and \( V_{ni,t} \) cannot cross at any date in between. Intuitively this is true because \( V_t(G) \) increases faster as time progresses because a larger reward is on the horizon.

The proof that \( g^* \) is optimal follows the lines from the binary example. By construction \( g^* \) provides the agent with exactly his no-information value. It is also efficient, implying that any schedule which yields a higher payoff for the principal cannot be individually rational for the agent. Here is a brief sketch of the proof that \( g^* \) is efficient. To incentivize effort the principal has but two instruments at his disposal. First the design of the schedule allocates effort across different realizations of the threshold. Any given amount of effort is least costly for the agent, and hence most efficient, when it is allocated in a way that maximizes the probability of surpassing the threshold. With the schedule \( g^* \), the agent completes the task with probability 1. However, \( g^* \) also induces the agent to spend additional, wasteful, effort that does not lead to additional rewards.
and indeed delays them. This is most efficiently incentivized by the second incentive instrument: the promise of valuable future disclosures. The schedule \( g^* \) delays any disclosure until after the agent has completed the initial \( t^* \) period of effort.

### 4.3 Moving The Goalposts

When \( V(g^\infty) \) is negative, it is no longer individually rational for the agent to follow a schedule such as \( g^* \) in which he completes the task with probability 1. However, the schedule \( V(g^\tau) \) is always implementable: it is the schedule in which the agent exactly completes all individually rational tasks. Indeed, as \( t \) ranges from \( \bar{\tau} \) to infinity, the payoff \( V(g^t) \) is continuous and strictly decreasing (as more and more non-individually rational tasks are added to the schedule). Thus, there exists a finite \( t^{**} > \bar{\tau} \) such that \( V(g^{t^{**}}) = 0 \).

Consider the following information policy. Before any effort decision the principal informs the agent whether the task is difficult \( (x > t^{**}) \) or easy \( (x \leq t^{**}) \). If the task is difficult the agent quits immediately. If the task is easy the agent commences working and at each future instant is informed only when he has completed the task. We show now that in response to this information policy the agent continues working until he completes the task and that the associated effort schedule

\[
g^{**}(x) = \begin{cases} 
x, & \text{if } x \leq t^{**} \\
0, & \text{otherwise}
\end{cases}
\]

is optimal for the principal. We refer to this policy as moving the goalposts because of the way the principal uses it to shift around the agent’s expectations of the difficulty of the task. It first drops expectations discretely (from \( E_F(x) \) to \( E_F(x \mid x \leq t^{**}) \)) when the principal suggests that the task is easy. The purpose is to encourage the agent to embark with the task rather than walk away as he would do in the absence of this disclosure. Indeed, it ultimately induces the agent to complete tasks even more difficult than the maximum individually rational task \( (t^{**} > \bar{\tau}) \). But once the agent begins working his expectations become more and more pessimistic as the principal repeatedly breaks the bad news that the task is not yet complete (i.e. \( E_F(x \mid t < x \leq t^{**}) \)).

**Proposition 6.** When \( g^\infty \) is not implementable, there exists a \( t^{**} > \bar{\tau} \) such that the schedule \( g^{**} \) is implementable and optimal and is achieved by the policy of moving the goalposts.
5 Designing the Task

The binary example in Section 3.1 shows that the principal benefits from a random threshold. In particular, by “splitting” a deterministic threshold into an easier task and a harder task, the principal can extract more effort on average from the agent. This requires that the incremental effort is individually rational so that the agent can be persuaded to keep working when the task turns out to be the harder one. Taking this one step further, each of these extreme tasks can be further split, resulting in a sequence of individually rational increments leading to higher overall expected effort. In this section we show how to exploit this possibility to the extreme by designing the optimal distribution over task difficulties.

When the principal can design the distribution of thresholds, she has a two-stage optimization. First she chooses the task distribution \( F \) and then she chooses the optimal information policy, leading to the induced schedule \( G \). However, we can simplify the problem by focusing on direct direct schedules, i.e. effort schedules \( G \) in which with probability 1 the agent works just long enough to complete the task, \( G(\{x, \tau : x = \tau\}) = 1 \).

**Lemma 3.** It is without loss of generality to optimize over direct schedules. A direct schedule is implementable if and only if \( V_t(G) \geq 0 \) at all dates \( t \geq 0 \) such that \( 1 - F(t) > 0 \).

The following lemma formalizes the key intuition behind the design of random tasks. A direct schedule is a lottery over quitting times. Due to discounting and costly effort the agent is risk-loving with respect to these lotteries. Intuitively the agent is willing to increase the difficulty of already difficult tasks in exchange for reducing the difficulty of tasks that are already easy. On the other hand, the principal values effort and she therefore has the opposite preferences. She is effectively risk averse. By building riskier lotteries into the schedule the principal can increase the agent’s value, then extract that value by asking the agent to work longer on average.

**Lemma 4.** Suppose that \( G \) is an individually rational schedule assigning positive mass to an interval \((x_l, x_h)\). Then there exists an individually rational schedule \( H \) in which all of the mass in the interval is moved to the endpoints and which is better for the principal, \( W(H) \geq W(G) \), strictly so when \( r_p < r \).

The lemma states that it is always possible to increase the principal’s payoff by adding risk to the schedule without violating individual rationality for the agent: the

\[\text{Equivalently, fixing the pure schedule } g^\infty \text{ and optimizing over } F.\]
agent will still be willing to begin working. However, if the resulting increment between successive tasks is too large, the schedule will not be implementable: it may not be possible to persuade the agent to continue. Thus it is the constraint of implementability that limits the extent to which Lemma 4 can be exploited and ensures that an optimal schedule exists.\footnote{The complication is that the implementability constraint is a conditional expectation which is not continuous in $G$, raising doubts about compactness. However we can show that if $G_k \to G$ (in the weak topology) and $V_t(G) < 0$ then for some $k$ there must be some (possibly different) time $s$ such that $V_s(G_k) < 0$, so that the set of schedules satisfying $V_t(G) \geq 0$ at all dates $t \geq 0$ is indeed compact.}

**Lemma 5.** An optimal direct schedule exists.

Having established existence we can complete the derivation of the optimal schedule using necessary conditions. For a distribution of task difficulties to be optimal it must not be possible to further increase risk without violating implementability. We show that this implies that the agent’s continuation value must be identically zero at every point in time. In particular, the task distribution cannot have atoms, it must have unbounded support, and it must have a constant hazard rate. These conditions pin down the distribution exactly.

**Proposition 7.** The optimal distribution over task difficulties is exponential with hazard rate \( r_c/R \).

## 6 Extensions

### 6.1 Alternative Formulations

**Uncertain Productivity** The results in our main model carry over if the threshold is fixed and the agent’s productivity is uncertain. Suppose output $y_t$ accumulates according to $y_t = \mu t$. The agent gets the reward if he quits at a time $\tau$ for which $y_\tau$ exceeds a fixed value $\bar{y}$. The agent does not know his productivity $\mu$ and does not observe the output. The principal knows the productivity,\footnote{Whether she observes output is irrelevant.} wants to maximize $1 - e^{-r\tau}$ and can provide information to the agent. If the agent quits at time $\tau$, his realized value is

\[
e^{-r\tau} 1_{y_\tau \geq \bar{y}} R - c (1 - e^{-r\tau}) .
\]
Defining $x = \frac{1}{\mu}$, we see that this model is equivalent to the one we have studied, provided that the distribution of $\frac{1}{\mu}$ satisfies Assumptions (A1) and (A2). Then all our results carry over.

Moving the goalposts has a new interpretation now. The principal first informs the agent that his productivity is high and as time progresses, the agent gradually leans that he is less productive.

**Shared Reward** If principal and agent share the reward, our results remain, largely, unchanged. Suppose the agent receives $\beta R$ when he stops past the threshold and the principal receives $(1 - \beta) R$. She still benefits from additional effort by receiving a flow payoff of $r$ when the agent works.

If the agent’s value under schedule $g^\infty$ is negative, the principal will use moving goalposts and have the agent stop exactly at $x$. Intuitively, if it is not optimal for her to let the agent work past the threshold when $\beta$ equals one, it cannot be optimal for $\beta < 1$.

If the agent’s value under $g^\infty$ is positive, the leading on policy in Proposition 5 remains optimal whenever

$$(1 - \beta) R \leq 1,$$

because the principal always prefers the agent to work past the threshold. If the inequality is reversed, both principal and agent prefer to stop exactly at $x$. Then, $g^\infty$ is optimal and leaves the agent with a positive rent.

### 6.2 Random Breakthroughs

In many settings, neither the principal nor the agent know how much work is required, because breakthroughs arrive randomly. A policy similar to moving the goalposts remains optimal in such a model. The principal initially provides just enough information to make the agent start working. Then, she slowly reveals the truth to keep him from stopping.

Specifically, the agent operates a Poisson bandit at flow cost $rc$ with uncertain arrival rate, which is either 0 (the project is bad) or $\lambda > 0$ (the project is good). His prior belief that the project is good is $\mu_0$. He can quit at any time to receive an outside option of zero. When the first Poisson increment realizes, he receives the reward $R$ and the game ends. We assume that

$$\lambda R > rc > 0,$$  \hspace{1cm} (A3)
so exerting effort for the agent is only optimal when the project is good. As before, the principal’s realized payoff if the game ends at time $t$ is $1 - e^{-rt}$.

The key difference to the model in Section 4 is that as time passes the agent becomes more pessimistic about receiving the reward and therefore is less willing to continue working. In the absence of a breakthrough, his belief evolves as

$$d\mu_t = -\lambda \mu_t (1 - \mu_t) dt. \quad (3)$$

Without information from the principal, the agent stops when his belief reaches a lower bound $\bar{\mu}$.

The principal can eliminate this negative drift by providing information. Suppose she provides “bad news” at constant rate $\lambda$ when the project is bad. This corresponds to a message process $m_t \in \{0, \lambda\}$ that satisfies

$$m_t = \lambda - \lambda N_t, \quad (4)$$

where $N_t$ is a second Poisson process, independent of the one driving the breakthroughs. It realizes with rate $\lambda$ if the project is bad and rate 0 when the project is good. In the absence of bad news the agent’s belief drifts upwards, which counters the negative drift when there are no breakthroughs. The agent then potentially works forever.

**Proposition 8.** If $\mu_0 \leq \bar{\mu}$, then the following policy is optimal. At time 0, the principal sends message $m_0 \in \{0, \lambda\}$ with probabilities

\[
\begin{array}{c|c|c}
  m=0 & 0 & \lambda \\
  m=\lambda & \alpha^* & 1 \\
\end{array}
\]

where $\alpha^* = \frac{\bar{\mu} - \mu_0}{\bar{\mu}(1 - \mu_0)}$. For any $t > 0$, she reveals information according to policy $m_t$ in Equation 4.

### 6.3 Preemption with Two Agents

Policies similar to leading on and moving goalposts can be optimal when the principal faces multiple agents with different preferences over task completion. We show this in a game of preemption between an upstart and an incumbent.

Both agents exert effort continuously at cost $rc$. The task accumulates output as long as both agents work and stops accumulating when one agent quits. The rewards to the
agents are asymmetric. If the task stops at time $\tau < x$, then both agents get zero, as before. If the task stops after $x$, but before $x + \delta$, the incumbent gets the reward $R$ and the upstart gets nothing. If the task stops after $x + \delta$, then the payoffs are reversed.

We can think of the agents as two collaborators on an engineering project who learn on the job and compete for a promotion. The upstart leans faster, but the incumbent starts out at a higher level of competence. If the task is completed at an intermediate stage, the incumbent’s initial advantage ensures he receives the promotion. However, with enough work, the upstart eventually catches up.

We assume that the upstart prefers to work for additional time $\delta$ if he can get the reward with certainty, i.e.

$$e^{-\tau \delta} R - c \left(1 - e^{-\tau \delta}\right) > 0.$$  \hspace{1cm} (A3)

For simplicity, we also assume that $x$ is distributed exponentially with parameter $\lambda$ and that the principal’s information is public.

Because the incumbent only receives the reward for intermediate values of $\tau$ for a given $x$, he has an incentive to preempt the upstart. Without information, this leads to a complete breakdown.

**Lemma 6.** Without information, the project stops at $\tau = 0$.

By providing information, the principal can improve on this outcome. If she stops at $x$ (or $x + \delta$) with certainty, then the upstart (or incumbent) will never receive the reward and refuse to participate. To ensure the schedule is implementable, she therefore must induce uncertainty about which agent receives the reward. This is achieved by the following extension of $g^\infty$:

$$G_p(\tau|x) = \begin{cases} x & \text{w. Pr. } p \\ x + \delta & \text{w. Pr. } 1 - p \end{cases}.$$  

This schedule randomizes between stopping at $x$ and $x + \delta$ with some probability $p$, which is chosen ex ante. Essentially, the principal is randomizing between who “wins” and always gives the winning agent the reward at the earliest possible time.

As with a single agent, the ex-ante values under $G_p$ may be positive or negative. The optimal policy sets $p$ to equalize both agents’ values.\footnote{The exponential distribution satisfies Assumptions (A1) and (A2). If we had a single agent, all results from Section 4 would go through.} If the values under $G_p$\footnote{The upstart’s value is decreasing in $p$ because of Assumption (A3), while the incumbent’s is increasing.}
cross above zero, the schedule leaves the agents with positive rents and the principal can use leading on to extract them. She asks the agents to work for a time \( t^* \) and then randomizes between stopping at \( x \) or \( x + \delta \). Now, the probability that the threshold has been passed is positive at any \( t > \delta \), which may lead the incumbent to quit. This policy is feasible only if the agents’ preferences are relatively aligned, i.e. \( \delta \) is small.

**Proposition 9.** Suppose for any \( \delta \) sufficiently small, the upstart’s and incumbent’s values under \( G_p \) cross strictly above zero. Then, for any such \( \delta \) there exist a probability \( p^* \) and a time \( t^* \) for which

\[
G^*(\tau|x) = \begin{cases} 
  t^* & \text{if } x \leq t^* \\
  x & \text{w. Pr. } p^* \\
  x + \delta & \text{w. Pr. } 1 - p^* 
\end{cases}
\]

is optimal and implementable.

If the agent’s values under \( G_p \) cross below zero, the principal has to use moving goalposts.

**Proposition 10.** Suppose that for all \( \delta \) sufficiently small, the upstart’s and incumbent’s values under \( G_p \) cross below zero. Then, for all \( \epsilon \) sufficiently small, there exists a small \( \delta \), a probability \( p^* \), and a time \( t^* \), such that

\[
G^*(\tau|x) = \begin{cases} 
  x & \text{if } x \leq t^* + \delta \\
  0 & \text{if } x > t^* + \delta \\
  x + \delta & \text{if } x \leq t^* \\
  0 & \text{if } x > t^* 
\end{cases}
\]

is implementable and there exists no other implementable schedule that leaves the principal with additional payoff higher than \( \epsilon / c \).

The principal instructs the agents to stop immediately when the threshold is above \( t^* \) (when the upstart receives \( R \)) or \( t^* + \delta \) (when the incumbent receives \( R \)). Otherwise she tells them to work and informs them only when to quit. The quitting time must be random to ensure implementability. If, say, she informs the agents that they have to work at most \( t^* \) with certainty, then the upstart can never receive the reward when the threshold is larger than \( t^* - \delta \), so he will prefer to quit once time \( t^* - \delta \) is passed. Under policy \( G^* \), the likelihood that either agent receives the reward for \( t \in (t^* - \delta, t^*) \)
is positive. For small $\delta$, this is sufficient to motivate both to keep working, provided we leave the agents with a small positive rent ex-ante.

References


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A Appendix

A.1 Proofs for Section 3

For the binary example we prove the results for the case of equal discounting $r = r_p$ because the specific method of proof previews the techniques for the general model of Section 4 where equal discounting is assumed. When $r_p < r$ we can use splitting arguments such as in the proof of Proposition 4 and Lemma 4 to show that an optimal policy must induce effort levels in $\{0, x_l\}$ and then proceed with the arguments below.

Proof of Proposition 1. This is a standard static Bayesian persuasion problem (see Kamenica and Gentzkow (2011) and Aumann, Maschler and Stearns (1995)) with effectively two actions for the agent, $\tau = 0$ and $\tau = x_l$. No other effort level would be optimal for the agent against any belief. In particular the agent’s optimal choice rule is $\tau = 0$ if $\mu < \bar{\mu}$ and $\tau = x_l$ if $\mu \geq \bar{\mu}$. The principal’s value function with respect to the agent’s belief is depicted in Figure 4 below in blue. The concavification is depicted in green.

![Figure 4: Static Bayesian Persuasion](image)

As is standard for these problems, the concavification illustrates the principal’s optimal value from maximizing the probability the agent chooses the principal’s preferred action, here $\tau = x_l$. When the prior exceeds $\bar{\mu}$ the agent chooses $x_l$ with probability 1 without any persuasion. When the prior is less than $\bar{\mu}$ the optimal persuasion sends
two messages inducing the two beliefs \( \mu = 0 \) and \( \mu = \bar{\mu} \), the latter message inducing the response \( \tau = x_l \).

**Proof of Proposition 2.** The mechanism described in the text leads to the following effort schedule, i.e. joint distribution over thresholds \( x \) and effort durations \( \tau \):

\[
\begin{array}{cccc}
& x = x_l & x = x_h \\
\tau = 0 & 0 & 0 \\
\tau = x_l & \alpha & 0 \\
\tau = x_h & \beta & 1 - \mu \\
\end{array}
\]

for some \( \alpha, \beta > 0 \) with \( \alpha + \beta = \mu \). Generally, a feasible effort schedule is some joint distribution whose total probability of \( x_l \) equals \( \mu \). Any implementable policy induces such a schedule (possibly involving more than just these three effort levels). Any schedule which is better for the principal must have a larger expected discounted effort duration. Since under the schedule above the agent completes the task with probability 1, increasing expected discounted effort only adds cost to the agent and delays rewards. Therefore the schedule above is efficient among all feasible schedules. Since it was constructed to yield zero expected payoff for the agent, no individually rational policy can improve for the principal.

**Proof of Proposition 3.** Moving the goalposts implements the following effort schedule:

\[
\begin{array}{cccc}
& x = x_l & x = x_h \\
\tau = 0 & 0 & \alpha \\
\tau = x_l & \mu & 0 \\
\tau = x_h & 0 & \beta \\
\end{array}
\]

where \( \alpha, \beta > 0, \alpha + \beta = 1 - \mu \).

To show that the schedule is efficient, it suffices to show that there is no alternative schedule with the same expected discounted effort but higher expected payoff for the agent. To see why, consider any schedule with a strictly greater payoff for the principal (i.e. expected discounted effort) and weakly higher expected payoff for the agent. Then there must be some excess effort: i.e. a positive probability of working for a positive duration not equal to the threshold. Eliminate enough excess effort to reduce the total expected effort to that of the moving the goalposts schedule, raising the agent’s expected payoff.

Now note that the implemented schedule completes the task with probability 1 conditional on the task being individually rational. This means that the effort is efficiently
allocated. That is, any alternative schedule with the same expected discounted effort must give the agent a lower expected payoff either because of delayed rewards or a lower probability of completing the task, or both. Thus, the schedule is efficient.

Moreover it gives the agent zero expected utility implying that the policy is optimal. This was already shown for priors greater than $\tilde{\mu}$. For priors less than $\tilde{\mu}$ note that the moving the goalposts mechanism leads to two possible initial beliefs: $\tilde{\mu}$ and 0. Quitting immediately is an optimal strategy for the agent at either of these beliefs and therefore the agent’s expected payoff is zero.

A.2 Proofs for Section 3.1

Proof of Proposition 4. When the threshold is randomized and full delayed disclosure is used, the agent’s expected payoff increases by

$$\left( c + R \right) \left[ \frac{e^{-r x_l} + e^{-r x_h}}{2} - e^{-r \tau} \right] := (c + R) \Delta_{a}^1,$$

whereas the principal’s payoff decreases by

$$\left[ 1 - e^{-r_p \tau} \right] - \left[ 1 - \frac{1}{2} \left( e^{-r_p x_l} + e^{-r_p x_h} \right) \right] = \frac{e^{-r_p x_l} + e^{-r_p x_h}}{2} - e^{-r_p \tau} := \Delta_{p}^1.$$

When partial delayed disclosure is added, the agent shifts effort probability from $x_l$ to $x_h$. Due to increased effort and delayed rewards, the agent’s payoff declines at rate

$$\left( c + R \right) \left[ e^{-r x_l} - e^{-r x_h} \right] := (c + R) \Delta_{a}^2.$$

Whereas the principal’s payoff increases at rate

$$\left( 1 - e^{-r_p x_h} \right) - \left( 1 - e^{-r_p x_l} \right) = e^{-r_p x_l} - e^{-r_p x_h} := \Delta_{p}^2.$$

Note that partial delayed disclosure ranges from full disclosure to no disclosure. Under the extreme of no disclosure the agent arrives at date $x_l$ assigning probability 1/2 to threshold $x_h$. By Equation 1 it would then be optimal for the agent to continue working (with probability 1) to $x_h$. Since $x_h$ is not individually rational this would give the agent a negative ex ante payoff. Thus, in the range between full delayed disclosure and no disclosure there exists a partial delayed disclosure policy which gives the agent exactly
zero ex ante payoff. See point C in Figure 3.

Therefore, to prove the Proposition it is enough to show that that \( \Delta_p^1 / (c + R) \Delta_a^1 \leq \Delta_p^2 / (c + R) \Delta_a^2 \), or equivalently that \( \Delta_p^1 / \Delta_p^2 \leq \Delta_a^1 / \Delta_a^2 \) with strict inequality when \( r_p < r \). For any discount rate \( r \), define

\[
\Delta^1 = \left( \frac{e^{-rx_l} + e^{-rx_h}}{2} \right) - e^{-r\tau} \\
\Delta^2 = e^{-rx_l} - e^{-rx_h}
\]

Using the definition of \( \tilde{\tau} \), these may be rewritten as follows

\[
\Delta^1 = e^{-r\tau} \left[ \left( \frac{e^{r\tau} + e^{-r\tau}}{2} \right) - 1 \right] \\
\Delta^2 = e^{-r\tau} \left( e^{r\tau} - e^{-r\tau} \right)
\]

so that the ratio equals

\[
\frac{\Delta^1}{\Delta^2} = \frac{(e^{r\tau} + e^{-r\tau}) - 2}{2(e^{r\tau} - e^{-r\tau})}.
\]

The derivative with respect to \( r \) has the same sign as

\[
(e^{r\tau} - e^{-r\tau})^2 - (e^{r\tau} + e^{-r\tau}) \left[ (e^{r\tau} + e^{-r\tau}) - 2 \right]
\]

or

\[
\left[ (e^{r\tau} - e^{-r\tau})^2 - (e^{r\tau} + e^{-r\tau})^2 \right] + 2(e^{r\tau} + e^{-r\tau}).
\]

Notice that the term in square brackets equals negative four, hence the derivative has the same sign as

\( (e^{r\tau} + e^{-r\tau}) - 2 \)

which is positive for all \( r > 0 \). This shows that \( \Delta_p^1 / \Delta_p^2 \leq \Delta_a^1 / \Delta_a^2 \) with a strict inequality when \( r_p < r \).

\[\square\]

B Proofs for Section 4

We begin with some preliminaries. Define for any \( t \),

\[
v(t) = v(t, t) = e^{-rt}R - c \left( 1 - e^{-rt} \right),
\]

\[32\]
i.e. the value of exactly completing the task when the level of difficulty is $t$. Note
that $v(t)$ is positive for $t = 0$, decreasing in $t$, and crosses zero at $\bar{t}$, the maximum
individually-rational threshold. We have for any $t$, including $t = \infty$,

$$V(g^t) = \frac{1}{F(t)} \left[ \int_0^t v(\hat{t}) f(\hat{t}) d\hat{t} \right].$$

As a function of $t$, $V(g^t)$ is strictly increasing up to $\bar{t}$ then decreasing. Similarly, for
$s < t$ the agent’s continuation value satisfies

$$V_s(g^t) = \frac{1}{F(t) - F(s)} \left[ \int_s^t v(\hat{t} - s) f(\hat{t}) d\hat{t} \right].$$

Lemma 7. $V_s(g^t)$ is weakly increasing in $s$.

Proof. First, consider $V_s(g^t)$ for some $s < t < \infty$. Taking derivatives,

$$\frac{\partial}{\partial s} V_s(g^t) = V_s(g^t) (r + H(s)) - (H(s) R - rc)$$

and

$$\frac{\partial^2}{\partial s^2} V_s(g^t) = \frac{\partial}{\partial s} V_s(g^t) (r + H(s)) - (R - V_s(g^t)) H'(s).$$

We know that $V_t(g^t) = R$ and that $V_s(g^t) < R$, since the agent incurs effort cost between
$s$ and $t$.

Suppose that $\frac{\partial}{\partial s} V_s(g^t) < 0$. Then, since $H$ is increasing by Assumption A2, $\frac{\partial^2}{\partial s^2} V_s(g^t) < 0$ and hence for all times $u \in [s, t]$, $V_u(g^t) < V_s(g^t) < 0$. But then $V_t(g^t) < V_s(g^t) < R$, which is a contradiction. Similarly, if $\frac{\partial}{\partial s} V_s(g^\infty) < 0$ then for any $u > s$

$$V_u(g^\infty) < (u - s) \frac{\partial}{\partial s} V_s(g^\infty) + V_s(g^\infty),$$

which crosses $-c$ as $u$ becomes sufficiently large. This is a contradiction since $V_u(g^\infty) \geq -c$, i.e. the agent cannot do worse than to work forever and never get the reward.\textsuperscript{13}

\textbf{Proof of Lemma 2.} Consider any date $u$. For any $u < \tau^0 \leq t$ we can decompose $V_u(g^t)$

\textsuperscript{13}Note here that we have normalized the flow cost of effort to be $rc$. The cost of working forever is thus $\frac{r}{\tau} = c$. 

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into

\[ V_u(g^t) = \int_u^{\tau^0} v(\hat{t})f(\hat{t})d\hat{t} + \frac{F(t) - F(\tau^0)}{F(t) - F(u)} V_0(g^t). \]

Suppose that at date \( u \) the agent’s optimal no-information continuation strategy is to work until \( \tau^0 \) and then quit. The first term above is the value of exactly completing the task when its difficulty is less than \( \tau^0 \). This is larger than the agent’s no-information continuation value \( V_{ni,u} \) since the latter completes the task with the same probability but later on average.

Thus, a sufficient condition for \( V_u(g^t) \geq V_{ni,u} \) is that \( V_0(g^t) \) is positive. If \( g^t \) is individually rational then \( V(g^t) = V_0(g^t) \geq V_{ni} \geq 0 \), so this follows directly from Lemma 7. The argument for \( V_s(g^\infty) \) is identical and hence omitted.

\[ \square \]

**Proof of Proposition 5.** Let \( t^* \) equate

\[ e^{-rt^*} [F(t^*)R + (1 - F(t^*)) V_t(g^\infty)] - c (1 - e^{-rt^*}) = V_{ni}, \]

and consider the pure schedule \( g^* \) defined in the statement of the Proposition.\(^{14}\) Note that \( t^* \) exceeds the no-information optimal effort \( \tau^0 \). This is because the left-hand side evaluated at any \( t^* \leq \tau^0 \) is strictly larger than \( V_{ni} \).

We first show that \( g^* \) is implementable. Note that after time \( t^* \), the policy \( g^* \) is identical to \( g^\infty \). Since \( g^\infty \) is implementable, \( V_t(g^*) \geq V_{ni,t} \) for all \( t \geq t^* \).

Next consider dates \( t \) earlier than \( t^* \), and note that \( g^* \) provides no information prior to \( t^* \). The agent’s no-information continuation value is therefore

\[ V_{ni,t} = \begin{cases} e^{-r(t^*-t)}F(\tau^0)R - c (1 - e^{-r(\tau^0-t)}) & \text{if } t \leq \tau^0 \\ F(t)R & \text{if } \tau^0 \leq t < t^* \end{cases} \]

because prior to \( \tau^0 \) the agent plans to continue working until \( \tau^0 \), and after \( \tau^0 \) the agent will quit immediately. (By Assumption (A1), past \( \tau^0 \) the marginal payoff from additional work is strictly negative.)

\(^{14}\)The LHS of the above equation is strictly decreasing in \( t \), so the value \( t^* \) at which it equals \( V_{ni} \) is unique. We can see this by rewriting it as

\[ (R + c) \left( e^{-rt}F(t) + \int_t^{\infty} e^{-rs}f(s) ds \right) - c \]

and calculating the derivative.
As for $V_t(g^*)$, the continuation value provided by the schedule $g^*$, we have for $t < t^*$

$$V_i(g^*) = e^{-r(t^*-t)} \left[ F(t^*)R + (1 - F(t^*))V_{t^*}(g^{\infty}) \right] - c \left( 1 - e^{-r(t^*-t)} \right).$$

The analysis now divides into two cases: when $\tau^0$ is positive and when $\tau^0$ is zero. First, suppose that $\tau^0 > 0$. Recall that $t^*$ is chosen such that $V(g^*) = V_{ni}$. For $t < \tau^0$, we calculate the following derivatives.

$$\frac{\partial}{\partial t} V_i(g^*) = r(V_i(g^*) + c)$$

and

$$\frac{\partial}{\partial t} V_{ni,t} = r(V_{ni,t} + c).$$

Since by construction, $V(g^*) = V_{ni}$, this ensures that $V_i(g^*) = V_{ni,t}$ for all $t \leq \tau^0$.

We now extend the argument to $t \geq \tau^0$. We can express $\frac{\partial}{\partial t} V_i(g^*)$ as

$$\frac{\partial}{\partial t} V_i(g^*) = re^{rt} (V(g^*) + c)$$

$$= re^{rt} (V_{ni} + c),$$

while the derivative of the no-information value is simply

$$\frac{\partial}{\partial t} V_{ni,t} = f(t)R.$$

From the definition of $V_{ni}$, it follows that

$$\frac{\partial}{\partial t} V_i(g^*) = re^{r(t-\tau^0)} \left( F(\tau^0)R + c \right).$$

By Assumption (A1), the no-information effort $\tau^0$ is the first time marginal value of additional effort is zero, i.e.

$$f(\tau^0)R - r \left( F(\tau^0)R + c \right) = 0.$$

Plugging this in, we obtain

$$\frac{\partial}{\partial t} V_i(g^*) = e^{r(t-\tau^0)} f(\tau^0)R.$$
Thus, \( \frac{\partial}{\partial t} V_t(g^*) \geq \frac{\partial}{\partial t} V_{ni,t} \) whenever

\[
e^{r(t-t^0)} f(t^0) \geq f(t).\]

This inequality holds because of Assumption (A1) and Grönwall’s Lemma.\(^{15}\) Therefore \( V_t(g^*) \geq V_{ni,t} \) on \( [t^0, t^*] \). Thus, \( g^* \) is implementable if \( t^0 > 0 \).

If \( t^0 = 0 \), we can use a similar proof as above. Specifically, we have

\[
V_{ni,t} = F(t)R
\]

for all \( t < t^* \) and as above

\[
\frac{\partial}{\partial t} V_t(g^*) = re^{rt} (V_{ni} + c) = re^{rt} c,
\]

because \( V_{ni} = 0 \). By Assumption (A1), we have

\[
\frac{\partial}{\partial t} V_{ni,t} = Rf(t) \leq Re^{rt} f(0).
\]

Note that \( t^0 = 0 \) only if \( rc > Rf(0) \) (the marginal value of effort at time zero is negative). We again have \( \frac{\partial}{\partial t} V_t(g^*) \geq \frac{\partial}{\partial t} V_{ni,t} \).

Having established that \( g^* \) is implementable, it remains to show that \( g^* \) is also optimal. Since \( g^* \) is exactly individually rational, we can establish that \( g^* \) is optimal by showing it is efficient. Suppose on the contrary that there was a schedule \( G \) such that \( W(G) > W(g^*) \) and \( V(G) \geq V(g^*) \). Since \( W(G) = E_G(1 - e^{-rt}) \) it must be that \( E_G e^{-rt} < E_{g^*} e^{-rt} \). Consider the pure schedule \( \tilde{g} \) defined by

\[
\tilde{g}(x) = g^*(x) + z
\]

where the constant \( z > 0 \) is chosen so that \( E_{\tilde{g}} e^{-rt} = E_G e^{-rt} \), and in particular \( W(\tilde{g}) = W(G) \). The new schedule \( \tilde{g} \) completes the task with probability 1 since \( g^* \) does and therefore

\[
V(\tilde{g}) = E_{\tilde{g}} [e^{-rt} R - c (1 - e^{-rt})] = E_G [e^{-rt} R - c (1 - e^{-rt})].
\]

\(^{15}\)See e.g. Hartman (2002), p.24. The Lemma states that whenever \( f'(s) \leq rf(s) \) on some interval \( [a, b] \), then \( f(s) \leq f(a)e^{rs} \). Taking \( a = t^0 \) and \( s = t - t^0 \) yields the result.
Moreover,

\[ V(G) = \mathbf{E}_G \left[ 1_{t \geq x} e^{-rt} R - c \left(1 - e^{-rt}\right) \right] \leq \mathbf{E}_G \left[ e^{-rt} R - c \left(1 - e^{-rt}\right) \right] \]

with equality if and only if \( G \) also completes the task with probability 1. It follows that \( G \) can dominate \( g^* \) only if \( G \) completes the task with probability 1. But then since \( \mathbf{E}_G e^{-rt} < \mathbf{E}_{g^*} e^{-rt} \)

\[ V(G) = \mathbf{E}_G \left[ e^{-rt} R - c \left(1 - e^{-rt}\right) \right] < \mathbf{E}_{g^*} \left[ e^{-rt} R - c \left(1 - e^{-rt}\right) \right] = V(g^*), \]

a contradiction.

\[ \square \]

**Proof of Proposition 6.** We begin by proving that \( g^{**} \) is implementable. Using Lemma 1 we know that it is sufficient for its continuation value to always exceed the continuation no-information value. Since \( t^{**} \) is chosen such that \( V(g^{**}) = V_{ni} \), the result follows from Lemma 2.

Next we prove that \( g^{**} \) is efficient which will establish that it is optimal (since it is exactly individually rational, i.e. \( V(g^{**}) = 0 \)). Note that for any schedule \( G \),

\[ V(G) = \mathbf{E}_G \left[ 1_{t \geq x} e^{-rt} R - c \left(1 - e^{-rt}\right) \right] \geq \mathbf{E}_G \left[ e^{-rx} R - c \left(1 - e^{-rt}\right) \right], \]

where \( 1_{t \geq x} \) is the indicator function for having completed the task. For the particular schedule \( g^{**} \), with probability one, either \( t = x \) or \( t = 0 \). Therefore

\[ V(g^{**}) = \mathbf{E}_{g^{**}} \left[ e^{-rx} R - c \left(1 - e^{-rt}\right) \right]. \]

Now if \( G \) is any schedule for which \( W(G) > W(g^{**}) \) then \( \mathbf{E}_G e^{-rt} < \mathbf{E}_{g^{**}} e^{-rt} \) and therefore

\[ V(G) \leq \mathbf{E}_G \left[ e^{-rx} R - c \left(1 - e^{-rt}\right) \right] < \mathbf{E}_{g^{**}} \left[ e^{-rx} R - c \left(1 - e^{-rt}\right) \right] = V(g^{**}). \]

This proves that \( g^{**} \) is efficient.

Finally, \( t^{**} > \bar{t} \) because \( V_t(g^{**}) \) is strictly increasing in \( t \) over \([0, \bar{t}]\) and \( V(g^{**}) = 0 \). Since \( t^{**} \) is defined by \( V(t^{**}) = 0 \) we must have \( t^{**} > \bar{t} \). \( \square \)
C Proofs for Section 5

Proof of Lemma 3. We prove the second claim first. The only-if part is obvious. Suppose that \( G \) is a direct schedule satisfying the conditions given. Consider the information policy consisting of two signals \( q \) (for “quit”) and \( s \) (for “stay”) such that at each date \( t \), the principal sends the signal \( q \) whenever \( x \leq t \) and otherwise the signal \( s \). The strategy for the agent of quitting immediately in response to \( q \) and otherwise continuing results in the schedule \( G \) and thus yields non-negative continuation value at every date. Indeed this strategy is optimal. There can be no improvement after receiving the signal \( q \), because conditional on \( q \) the agent knows that the task is complete. The only other deviation would be to quit after receiving \( s \) but this would yield a continuation payoff of zero which cannot improve upon the non-negative continuation value from continuing.

To prove the first claim, let \( \tilde{G} \) be any implementable schedule and consider the direct schedule \( G \) whose marginal over effort duration is the same as \( \tilde{G} \). Since the principal’s payoff depends only on effort we have \( W(G) = W(\tilde{G}) \). Moreover at every date \( t \)

\[
V_t(G) \geq V_t(\tilde{G}) \geq 0.
\]

We have the first inequality because \( G \) entails the same effort costs but provides the reward with probability 1 and with no delay. We have the second inequality because \( \tilde{G} \) is implementable and therefore provides at least the no-information continuation value at each date. It now follows that \( G \) is implementable.

We prove the following more specific version of Lemma 4.

**Lemma 8.** If \( G \) is a schedule assigning positive mass to an interval \( (x_l, x_h) \) then there exists another schedule \( H \) which is identical to \( G \) outside the interval \([x_l, x_h]\) and for which \( V(H) = V(G) \) and \( W(H) \geq W(G) \) with a strict inequality when \( r_p < r \).

**Proof.** The agent’s expected payoff from any schedule \( G \) is

\[
V(G) = E_G \left[ e^{-rt}R - c(1 - e^{-rt}) \right] = E_G \left[ e^{-rt} (R + c) \right] - c
\]

while the principal’s is

\[
W(G) = E_G (1 - e^{-rpt}).
\]
Applying positive affine transformations we can represent these preferences equivalently by

\[ \tilde{V}(G) = E_G \tilde{v}(t) \]
\[ \tilde{W}(G) = -E_G \tilde{w}(t) \]

where

\[ \tilde{v}(t) = \frac{e^{-rt} - e^{-rx_h}}{e^{-rx_l} - e^{-rx_h}} \]
\[ \tilde{w}(t) = \frac{e^{-r_p t} - e^{-r_p x_h}}{e^{-r_p x_l} - e^{-r_p x_h}}. \]

In particular \( \tilde{V}(G) \geq \tilde{V}(G') \) if and only if \( V(G) \geq V(G') \) and likewise for the principal.

Suppose \( G \) attaches positive probability to the interval \((x_l, x_h)\). Consider schedules that are identical to \( G \) except that all of the mass inside the interval \((x_l, x_h)\) is moved to atoms at the endpoints \(\{x_l, x_h\}\). Among such schedules, the one with all of the mass at \(x_h\) is strictly worse for the agent than \(G\) and the one with all of the mass at the low end is strictly better. Moreover the agent's payoff increases continuously as this mass moves from \(x_h\) to \(x_l\) and therefore there exists a schedule \(H\) such that the agent is indifferent between \(G\) and \(H\).

Similarly there exists among these a schedule \(J\) such that \(E_J = E_G\). We have the following identity.

\[
\tilde{V}(H) = \left[ \tilde{V}(H) - \tilde{V}(J) \right] + \tilde{V}(J) + \tilde{V}(G)
\]
\[
\tilde{W}(H) = \left[ \tilde{W}(H) - \tilde{W}(J) \right] + \tilde{W}(J) + \tilde{W}(G).
\]

We will show that because \(r_p \leq r\), \(\Delta_p^2 + \Delta_p^1 \geq \Delta_a^2 + \Delta_a^1\) and therefore since \(\tilde{V}(H) = \tilde{V}(G)\) we also have \(\tilde{W}(H) \geq \tilde{W}(G)\) and the inequality will be strict when \(r_p < r\).

Note that \(\Delta_a^1\) is positive because \(\tilde{\vartheta}\) is strictly convex. That means \(\Delta_a^2\) is negative. In particular \(\tilde{V}(H) < \tilde{V}(J)\). Since \(H\) and \(J\) differ only in how the mass is divided between the points \(x_l\) and \(x_h\), it follows that \(H\) has a larger mass at \(x_h\). It follows that \(\Delta_p^2\) is positive and indeed by our normalization of payoffs,

\[ \Delta_p^2 = -\Delta_a^2 \]

since according to the normalized payoff functions \(\tilde{\vartheta}\) and \(\tilde{\omega}\) any shift of mass from \(x_l\) to
\( x_h \) is a one-for-one transfer of utility from agent to principal.

Thus, to show that \( \tilde{W}(H) \geq \tilde{W}(G) \) it is now enough to show that
\[
|\Delta^1_p| \leq \Delta^1_r
\]

i.e. the principal’s loss from the mean-preserving spread is smaller than the agent’s gain with a strict inequality when \( r_p < r \). Expanding,
\[
|\tilde{W}(J) - \tilde{W}(G)| \leq \tilde{V}(J) - \tilde{V}(G)
\]
\[
|E_G \tilde{w} - E_J \tilde{w}| \leq E_J \tilde{\vartheta} - E_G \tilde{\vartheta}.
\]

Since \( \tilde{w} \) is convex, \( E_J \tilde{w} \geq E_G \tilde{w} \) and so the inequality is equivalent to
\[
E_J \tilde{w} - E_G \tilde{w} \leq E_J \tilde{\vartheta} - E_G \tilde{\vartheta}
\]
or
\[
E_J (\tilde{w} - \tilde{\vartheta}) \leq E_G (\tilde{w} - \tilde{\vartheta}).
\]

Since the schedules \( J \) and \( G \) are identical outside of the interval \([x_l, x_h]\), this reduces to
\[
E_J (\tilde{w} - \tilde{\vartheta} \mid x_l \leq x \leq x_h) \leq E_G (\tilde{w} - \tilde{\vartheta} \mid x_l \leq x \leq x_h).
\]

Furthermore, conditional on the interval \([x_l, x_h]\), the schedule \( J \) assigns all of its mass to the points \( x_l \) and \( x_h \) where the two functions \( \tilde{w} \) and \( \tilde{\vartheta} \) are equal. Thus, the left-hand side is zero and we need only to show that
\[
E_G (\tilde{w} - \tilde{\vartheta} \mid [x_l, x_h]) \geq 0.
\]

In fact \( \tilde{w} \) pointwise dominates \( \tilde{\vartheta} \) on the interval as we now show.

Since \( \tilde{w}(x_l) = \tilde{\vartheta}(x_l) \) and \( \tilde{w}(x_h) = \tilde{\vartheta}(x_h) \) and both functions are decreasing and continuously differentiable, we can show that \( \tilde{w} \geq \tilde{\vartheta} \) on the entire interval by showing that \( \tilde{w}'(\cdot) - \tilde{\vartheta}'(\cdot) \) is decreasing. Because if \( \tilde{\vartheta}(t) > \tilde{w}(t) \) for some \( t \in [x_l, x_h] \) then \( \tilde{\vartheta}'(s) \geq \tilde{w}'(s) \) for some \( s \leq t \) but if \( \tilde{w}'(\cdot) - \tilde{\vartheta}'(\cdot) \) is continuous and decreasing then \( \tilde{\vartheta}'(s') \geq \tilde{w}'(s') \) for all \( s' \geq t \) implying \( \tilde{\vartheta}(x_h) > \tilde{w}(x_h) \), a contradiction.

Now \( \tilde{w}'(\cdot) - \tilde{\vartheta}'(\cdot) \) is decreasing if \( \tilde{w}''(\cdot) \leq \tilde{\vartheta}''(\cdot) \) for all \( t \in [x_l, x_h] \) and this inequality follows immediately because \( r \geq r_p \) and the inequality is strict when \( r > r_p \). \qed
Proof of Lemma 5. A direct schedule has $x = \tau$ with probability 1, therefore the set of all direct schedules can be identified with the set of probability measures on the non-negative real numbers. The principal’s value for a direct schedule is $W(G)$ which, viewed as function of $G$, is continuous in the weak topology. An optimum exists provided the feasible set, i.e. the subset of implementable schedules, is compact in that topology.

A schedule is implementable only if it is individually rational $V(G) \geq 0$. We begin by showing that the subset of individually rational schedules $G$ is compact. Consider any $\varepsilon > 0$. Define $t(\varepsilon)$ as follows

$$R + \varepsilon \cdot e^{-rt(\varepsilon)}v(t(\varepsilon)) = 0.$$

There is no individually rational schedule assigning greater than $\varepsilon$ probability to thresholds $t(\varepsilon)$ or higher. This shows that the set of individually rational schedules is tight and therefore relatively compact.\(^\text{16}\) It is also closed (and hence compact) as the set of measures giving a non-negative expected value of a continuous payoff function.

To show that the subset of implementable schedules is compact it is now enough to show that it is closed. Consider a sequence $G_k$ of implementable schedules converging to a schedule $G$. Let $t$ be any date such that $1 - G(t) > 0$. Then also\(^\text{17}\) for all $k$ sufficiently large and for all $\varepsilon > 0$, sufficiently small, $1 - G_k(t + \varepsilon) > 0$. Since $G_k$ is implementable, $V_{t+\varepsilon}(G_k) \geq 0$. Since $V_{t+\varepsilon}(G_k) = E_{G_k}(v(x) | x > t + \varepsilon)$ we have\(^\text{18}\)

$$(1 - G_k(t + \varepsilon))V_{t+\varepsilon}(G_k) + G_k(\{t + \varepsilon\}) \cdot R = \int_{t+\varepsilon}^{\infty} v(s-t-\varepsilon)dG_k(s)$$

and therefore the right-hand side is non-negative. Moreover, the integral of a continuous function over a closed interval is upper-semicontinuous\(^\text{19}\) in $G_k$ and thus

$$\int_{t+\varepsilon}^{\infty} v(s-t-\varepsilon)dG(s) \geq 0.$$

\(^\text{16}\)See (Aliprantis and Border, 1999, Section 12.5).
\(^\text{17}\)See (Aliprantis and Border, 1999, Section 14.5)
\(^\text{18}\)Recall that we have defined the notation $v(x) = e^{-rx}R - c(1 - e^{-rx})$ as the payoff from exactly completing the task. Also, $G_k(\{t + \varepsilon\})$ is the mass of a potential atom at $t + \varepsilon$.
\(^\text{19}\)See (Aliprantis and Border, 1999, Section 14.5)
Now
\[ V_t(G) \geq \frac{1}{1 - G(t)} \left[ \int_{t+\varepsilon}^{\infty} e^{-r(s-t)} R - c(1 - e^{-r(s-t-\varepsilon)}) dG(s) - c(1 - e^{-r\varepsilon}) \right] \]
or
\[ V_t(G) \geq \frac{1}{1 - G(t)} \int_{t+\varepsilon}^{\infty} v(s - t - \varepsilon) dG(s) - \delta(\varepsilon) \]
where \( \delta(\varepsilon) = c(1 - e^{-r\varepsilon}) + R \int_{t+\varepsilon}^{\infty} e^{-r(s-t-\varepsilon)} - e^{-r(s-t)} dG(s) \). Since the integral is non-negative,
\[ V_t(G) \geq -\delta(\varepsilon). \]

This is true for all sufficiently small \( \varepsilon \) and moreover \( \delta(\varepsilon) \) vanishes as \( \varepsilon \) does, hence \( V_t(G) \geq 0 \) for all \( t \) such that \( 1 - G(t) > 0 \) proving that \( G \) is implementable and the set of implementable schedules is closed and hence compact. \( \square \)

**Lemma 9.** An optimal schedule has no atoms.

**Proof.** Suppose \( G \) is implementable and has an atom at time \( t \). For any \( \varepsilon > 0 \), the continuation value at time \( t - \varepsilon \) is at least
\[ \delta e^{-r\varepsilon} R - c(1 - e^{-r\varepsilon}) \]
where \( \delta \) is the size of the atom. This is bounded away from zero for \( \varepsilon \) small enough. Pick such an \( \varepsilon \) and consider the interval \([t - \varepsilon, t + \varepsilon]\). We split a fraction \( v \) of the atom between \( t - \varepsilon \) and \( t + \varepsilon \) so that we make the agent indifferent at \( t - \varepsilon \). That is, if \( vq \) is the fraction of mass moved to \( t - \varepsilon \), we pick \( q \) such that
\[ e^{-r\varepsilon} R - c(1 - e^{-r\varepsilon}) = qR + (1 - q) \left( e^{-r2\varepsilon} R - c \left( 1 - e^{-r2\varepsilon} \right) \right). \]
At this \( q \), the principal is weakly better off at \( t - \varepsilon \). This is because the above equation is equivalent to
\[ 1 - e^{-r\varepsilon} = (1 - q) \left( 1 - e^{-r2\varepsilon} \right) \]
so that
\[ 1 - q = \frac{1 - e^{-r\varepsilon}}{1 - e^{-r2\varepsilon}}. \]
The change in the principal’s value is then proportional to
\[ (1 - q) \left( 1 - e^{-r2\varepsilon} \right) - (1 - e^{-r\varepsilon}) \].

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Plugging in the expression for $q$ we can see that this value is positive because the function $(1 - e^{-r\varepsilon}) / (1 - e^{-r^2\varepsilon})$ is increasing in $r$ for sufficiently small $\varepsilon$.

For small enough $\nu > 0$ and $\varepsilon > 0$ the new schedule $G'$ is implementable. For dates $s < t - \varepsilon$ this follows because $G'$ is identical to $G$ prior to $t - \varepsilon$. Any change after $t - \varepsilon$ which leaves the continuation value unchanged at $t - \varepsilon$ must also leave the continuation value unchanged at prior dates. For dates $s \geq t + \varepsilon$ the continuation schedule under $G'$ is the same as under $G$ and since $G$ was implementable, we have $V_s(G') \geq 0$.

Finally for $s \in (t - \varepsilon, t + \varepsilon)$, the agent’s value under $G'$ is bounded from below by

$$v(1 - q)e^{-r^2\varepsilon}R - c \left(1 - e^{-r^2\varepsilon}\right).$$

This value is positive for $\varepsilon$ sufficiently small. Hence $G'$ is implementable.

\[ \square \]

**Lemma 10.** Suppose $G$ has no atoms and $V_t(G) > 0$ for some $t$. Then there exists $B > 0$ such that for all $\varepsilon > 0$ for all sufficiently small intervals $[t, u]$ we have

1. $V_s(G) > B$ for all $s \in [t, u]$
2. $\frac{1 - G(u)}{1 - G(t)} > 1 - \varepsilon$.

**Proof.** Since $G$ is atomless, it is continuous, which implies that $V_t(G)$ is continuous at $t$. The first point in the Lemma then follows from continuity of $V_t(G)$, and the second from continuity of $G$ and the fact that $1 - G(u) \leq 1 - G(t)$.

\[ \square \]

**Lemma 11.** An optimal schedule must have $V_t(G) = 0$ for all $t \geq 0$.

**Proof.** By Lemma 9 an optimal schedule $G$ has no atoms. Suppose $V_t(G) > 0$ for some $t$. Then consider an interval $[t, u]$ and the schedule $H$ obtained from $G$ by applying Lemma 4. Consider any $s \in [t, u]$. When the interval is small enough,

$$V_s(H) \geq \left(\frac{1 - H(u)}{1 - H(s)}\right)e^{-r(u-t)}V_u(H).$$

The continuation value $V_u(H)$ is equal to $V_u(G)$ because $H$ and $G$ are identical after $u$. And since $V_s(G)$ cannot be larger than $R$,

$$V_s(G) - V_s(H) \leq R \left(1 - \frac{1 - H(u)}{1 - H(s)}\right).$$
Now
\[ \frac{1 - H(u)}{1 - H(s)} = \frac{1 - H(u)}{1 - H(t)} \geq \frac{1 - G(u)}{1 - G(t)} \]
where the equality follows from the fact that \( H \) has no mass in the interval \((t, u)\) and the inequality from the fact that \( H \) was obtained from \( G \) by moving mass in that interval to the endpoints.

We now apply Lemma 10 to choose for any \( \varepsilon > 0 \), the endpoint \( u \) close enough to \( t \) to obtain the bounds \( V_s(G) > B > 0 \) for all \( s \in [t, u] \) and
\[ V_s(G) - V_s(H) \leq \varepsilon R. \]
Therefore the schedule \( H \) has a positive continuation value on the interval \([t, u]\). The continuation value at later dates is the same as under the original schedule \( G \) and therefore non-negative. The continuation value at earlier dates is the same as that of \( G \) since \( V(H) = V(G) \) and \( H \) is identical to \( G \) at all dates earlier than \( t \).

The schedule \( H \) is thus implementable and strictly better for the principal, hence \( G \) cannot be optimal. \( \square \)

**Proof of Proposition 7.** If \( G \) is exponential with parameter \( \lambda \), then the agent’s continuation value is constant and equal to
\[ V_t = \frac{\lambda}{r + \lambda} R - \frac{r}{r + \lambda} c \]
at all times. Choosing \( \lambda \) so that \( \lambda R = rc \) implies that the continuation value is zero for all \( t \geq 0 \). Hence the exponential distribution achieves the optimum. \( \square \)

### D Proofs for Section 6.2

**Proof of Proposition 8.** Without information, the agent solves an optimal stopping problem. His value is Markovian in the belief and is given by the HJB equation
\[ rV_{ni}(\mu) = -rc + \mu \lambda (R - V_{ni}(\mu)) - V'_{ni}(\mu) \lambda \mu (1 - \mu) \]
with boundary conditions \( V_{ni}(\bar{\mu}) = 0 \) and \( V'_{ni}(\bar{\mu}) = 0 \). The first boundary condition implies that \( \bar{\mu} \) solves
\[ \bar{\mu} \lambda R = rc. \]
His value under policy $m_t$ is stationary. To see this, we can calculate the evolution of the belief for a small interval of time $[t, t + h]$ without any news or breakthroughs, which is

$$
\mu_{t+h} = \frac{e^{-\lambda h} \mu_t}{e^{-\lambda h} \mu_t + (1 - \mu_t)(1 - \alpha)}.
$$

Here $\alpha$ is the probability that bad news realizes on $[t, t + h]$, conditional on the project being bad. Since $N_t$ has arrival rate $\lambda$ we have $\alpha = 1 - e^{-\lambda h}$, which yields $\mu_{t+h} = \mu_t$.

With a stationary belief, the agent’s value becomes

$$
\hat{V}(\mu) = \frac{\mu \lambda R - rc}{r + \lambda}.
$$

This value is positive whenever $\mu \geq \bar{\mu}$ and negative otherwise. At belief $\bar{\mu}$, the agent’s no-information value is zero, and so is $\hat{V}$. Thus, he is indifferent between following this policy and quitting. This shows that the policy is feasible.

We now prove optimality. Any optimal policy must have the agent receive the reward with probability one if the project is good. To see this, suppose that the agent quits at time $t$ but the project is good. Then, providing full information at time $t$ is a Pareto improvement. Thus, optimal policies can only differ in the expected effort of the agent when the project is bad.

Whenever the agent’s initial belief is below $\bar{\mu}$, the principal can get the agent to start working under policy $m_t$ only if she induces a belief above $\bar{\mu}$. The initial message $m_0$ in the proposition statement is the concavification of the principal’s value. It maximizes the likelihood that the agent starts working by inducing a belief of exactly $\bar{\mu}$. Otherwise, it induces belief $0$. Because of this, it holds the agent exactly to an outside value of zero. Also, our policy always has the agent working until he receives the reward if the project is good.

We can show that any other schedule which provides a higher value for the principal than our policy must have the agent more effort on average when the project is bad, which then violates his participation constraint. Thus our policy is optimal.\(^{20}\)

\(^{20}\)The argument is analogous to the one in Proposition 5.
E Proofs for Section 6.3

Proof of Lemma 6. The no-information value of the incumbent is

\[ V_{ni}^I = (F(\tau) - F(\tau - \delta)) e^{-r \tau} R - c \left( 1 - e^{-r \tau} \right). \]

For any time \( \tau \geq \delta \), his likelihood of obtaining the reward is decreasing in \( \tau \).\(^{21}\) Thus, the incumbent never works past time \( \delta \). But this means that the upstart never receives the reward, so he will quit immediately. The project thus stops at \( \tau = 0 \). \( \square \)

Proof of Proposition 9. The realized payoffs of the upstart and incumbent are

\[ u^U(\tau, x) = \begin{cases} -c(1 - e^{-r\tau}) & \text{if } \tau < x + \delta \\ e^{-r\tau}R - c(1 - e^{-r\tau}) & \text{if } \tau \geq x + \delta \end{cases} \]

and

\[ u^I(\tau, x) = \begin{cases} -c(1 - e^{-r\tau}) & \text{if } \tau < x \text{ or } \tau \geq x + \delta \\ e^{-r\tau}R - c(1 - e^{-r\tau}) & \text{if } x < \tau < x + \delta. \end{cases} \]

We can show that the upstart’s participation constraints are satisfied at all positive times whenever his ex-ante constraint is satisfied.\(^{22}\) We thus only focus on the incumbent. We first derive the incumbent’s no-information continuation value for arbitrary times \( t \) under schedule \( G_p \). We will use this value to establish that \( G^* \) is feasible for times past \( t^* \).

Under \( G_p \), the agents update their beliefs about whether the principal will stop at \( x \) or \( x + \delta \) as time passes. We can calculate their beliefs as

\[ \Pr(x \leq \hat{x}|t) = \begin{cases} \frac{F(\hat{x}) - (pF(t) + (1 - p)F(t - \delta))}{\Pr(t)} & \text{if } \hat{x} > t \\ \frac{F(\hat{x}) - F(t - \delta)}{(1 - p)\Pr(t)} & \text{if } t - \delta \leq \hat{x} \leq t. \end{cases} \] \( \quad (6) \)

Here,

\[ \Pr(t) = p(1 - F(t)) + (1 - p)(1 - F(t - \delta)) \]

\(^{21}\)The derivative is \( f(\tau) - f(\tau - \delta) = \lambda e^{-\lambda \tau} - \lambda e^{-\lambda(\tau - \delta)} \), which is negative.

\(^{22}\)Intuitively, the incumbent wants to deviate at whenever he expects that the threshold has already been passed. That probability is positive under \( G^* \). However, the probability that \( t^* + \delta \) has been passed is zero for all \( t > 0 \), so there are no such considerations for the upstart.
is the probability that the project has not stopped at time $t$. The preemption motive of the incumbent implies that for any time $t$ the task stops immediately without further information, so the no-information continuation effort is zero. The argument is similar to the one for Lemma 6 and we omit it here.

However, the incumbent’s no-information value is positive, because the likelihood that the threshold has already been reached is positive. It is given by

$$V_{ni,t}^I = \frac{(F(t) - F(t - \delta))(1 - p)R}{\Pr(t)}$$

and admits the closed-form expression

$$V_{ni,t}^I = \begin{cases} 
\frac{1 - e^{-\lambda t}}{pe^{-\lambda t} + (1 - p)}(1 - p)R & \text{if } t < \delta \\
\frac{1 - e^{-\lambda \delta}}{pe^{-\lambda \delta} + (1 - p)}(1 - p)R & \text{if } t \geq \delta 
\end{cases}$$

Using the updating formula in Equation 6, we can also derive the incumbent’s continuation value under $G_p$, which is

$$V_I^I(G_p) = \frac{pe^{-\lambda t}}{pe^{-\lambda t} + (1 - p)}V(I^\infty) - \frac{(1 - p)e^{-\lambda t}}{pe^{-\lambda t} + (1 - p)}e^r\left(1 - \frac{\lambda}{r + \lambda}e^{-r\delta}\right)$$

for $t \leq \delta$ and $V_I^I(G_p) = V_I^I(G_p)$ for $t > \delta$ because of the memorylessness property.

The ex-ante value $V_I^I(G_p)$ can be obtained by evaluating the expression above at $t = 0$. The upstart’s ex-ante value $V_U^I(G_p)$ can be obtained similarly. The agents’ ex-ante values are equal whenever

$$p = \frac{1}{1 + e^{-r\delta}}.$$  

We are now ready to prove implementability of $G_p$. If $t \geq \delta$, both the value under $G_p$ and the no-information value are stationary. The schedule is feasible at any such time whenever the incumbent’s value $V_I^I(G_p)$ exceeds the no-information value in Equation 7. By our assumption in the statement of the proposition, the incumbent’s ex-ante value is bounded above zero for any $\delta < \bar{\delta}$. That is, there exists some value $K > 0$ such that for all $\delta < \bar{\delta}$, we have

$$V_I^I(G_p) \geq K$$

whenever Equation 9 holds. Using the closed form for the incumbent’s ex-ante value.
(obtained from Equation 8) we can show that $V^I_i(G_p) \geq V^I_{m, t}$ when $p$ satisfies Equation 9 whenever

$$e^{-\lambda \delta} K \geq (1 - e^{-\lambda \delta}) \frac{e^{-r \delta}}{1 + e^{-r \delta}} R.$$  \hspace{1cm} (11)$$

This inequality holds for sufficiently small $\delta$. This shows that $G_p$ is implementable for all $t \geq \delta$.

It remains to show that the same holds for $t < \delta$. Plugging Equation 9 into the no-information value in Equation 7 and the closed form for $V^I_i(G_p)$ at time $t < \delta$, we can derive an inequality similar to Equation 11 and we can show that if Equation 11 holds, then that inequality is also satisfied. This concludes our argument that $G_p$ is implementable.

We can now establish implementability of $G^*$. Because the exponential distribution is memoryless, under schedule $G^*$ the values of the agents are

$$V^i_t(G^*) = \begin{cases} e^{-r(t^* - t)} V^i_t(G_p) - c \left(1 - e^{-r(t^* - t)} \right) & \text{for } t \leq t^* \\ V^i_{t-t^*}(G_p) & \text{for } t > t^* \end{cases},$$

where $i \in \{I, U\}$. The probability $p$ and time $t^*$ are chosen so that the ex-ante values $V^I(G^*)$ and $V^{UI}(G^*)$ are both equal to zero. For $t \leq t^*$, the no-information value of both agents is zero as well, because they haven’t received any information about $x$. \hspace{1cm} (23)

Since $V^I_t(G^*)$ and $V^{UI}_t(G^*)$ are both increasing in time for $t \leq t^*$ (because their continuation value at $t^*$ is positive) this establishes that $G^*$ does not violate any participation constraints on the time interval $[0, t^*]$.

For any time $t > t^*$ both agents’ continuation values under $G^*$ are the same as their values at time $t - t^*$ under schedule $G_p$. Their no-information continuation values are the same as well. Thus, showing that $G^*$ is implementable past $t^*$ is equivalent to showing $G_p$ is implementable at any time, which we already have done. Thus, $G^*$ is implementable.

The argument that it is optimal is the same as in the single-agent case in Proposition 5, noting that if the incumbent’s ex-ante participation constraint binds, any schedule that makes the principal better off must necessarily violate that participation constraint, since it induces more effort on average.

\[\square\]

\hspace{1cm} \hspace{1cm} 23The argument is the same we used in the proof of Proposition 5 with a single agent.
Proof of Proposition 10. As before, we only focus on the incumbent. The beliefs of the agents at time $t$, after the principal has asked them to start working are

$$\Pr(x \leq \hat{x}|t) = \Pr(x \leq \hat{x} \wedge \text{stop at } x|t) + \Pr(x \leq \hat{x} \wedge \text{stop at } x + \delta|t)$$

(12)

where

$$\Pr(x \leq \hat{x} \wedge \text{stop at } x|t) = \frac{(F(\hat{x}) - F(t))p}{\Pr(t)}$$

for $t \leq \hat{x} \leq t^* + \delta$ is the conditional distribution of $x$ at time $t$ when the principal stops at $x$ and

$$\Pr(x \leq \hat{x} \wedge \text{stop at } x + \delta|t) = \begin{cases} \frac{(F(\hat{x}) - F(t - \delta))(1 - p)}{\Pr(t)} & \text{if } t - \delta \leq \hat{x} \leq t^* \\ \frac{(F(t^*) - F(t - \delta))(1 - p)}{\Pr(t)} & \text{if } t^* \leq \hat{x} \leq t^* + \delta \end{cases}$$

is the conditional distribution when the principal stops at $x + \delta$. Here

$$\Pr(t) = p(F(t^* + \delta) - F(t)) + (1 - p)(F(t^*) - F(t - \delta))$$

is the likelihood that the task has not stopped yet given $p$ and $t^*$. They are derived using the same logic as for Equation 6.

Using these beliefs, we can calculate the incumbent’s no-information value. As in the leading on case, the no-information continuation effort is zero\textsuperscript{24} and the incumbent’s no-information value is given by

$$V_{ni,t}^{I} = \frac{\Pr(x \leq t \wedge \text{stop at } x + \delta|t)}{\Pr(t)} R.$$

It admits the closed form solution

$$V_{ni,t}^{I} = \begin{cases} \frac{(1 - e^{-\lambda t})(1 - p)R}{p(e^{-\lambda t} - e^{-\lambda(t^* + \delta)}) + (1 - p)(1 - e^{-\lambda t^*})} & \text{if } t < \delta \\ \frac{(1 - e^{-\lambda \delta})(1 - p)R}{(1 - e^{-\lambda \delta} + (1 - p))(1 - e^{-\lambda(t^* - (t - \delta))})} & \text{if } \delta \leq t \leq t^* \\ \frac{(1 - p)R}{pe^{-\lambda \delta} + (1 - p)} & \text{if } t^* < t \leq t^* + \delta. \end{cases}$$

\textsuperscript{24}The argument is analogous to the one in Proposition 9. We are therefore omitting a detailed derivation.
The continuation value under $G^*$ is

$$V^I_t(G^*) = \frac{p}{\Pr(t)} \int_t^{t^*+\delta} \left( e^{-r(x-t)} R - c \left( 1 - e^{-r(x-t)} \right) \right) f(x) dx$$

$$- \left( 1 - p \right) \frac{c}{\Pr(t)} \int_{\min(t-\delta,0)}^{t^*} \left( 1 - e^{-r(x+\delta-t)} \right) f(x) dx. \tag{14}$$

Using the values for the incumbent’s continuation and no-information values, we now show the following: for any small $\epsilon$, there exists a small $\delta$ such that the schedule $G^*$ with $p = 1/2 + \epsilon$ and some time $t^*$ is feasible for all $t$ and the upstart’s and incumbent’s values are between zero and $\epsilon$. This implies the statement in the proposition.

We can show that both $V^I$ and $V^U$ are decreasing in $t^*$. They reach $(1/2 + \epsilon)R$ and $(1/2 - \epsilon)R$ respectively when $t^*$ is zero, and by our assumption that under schedule $G_p$ the agents’ values cross below zero, they are both negative as $t^*$ becomes large. Thus, it is always possible to find a $t^*$ such that for $\delta$ sufficiently small, the both agents’ values are between zero and $\epsilon$.

Exploiting the memorylessness property, we can show that for all $t \leq t^* + \delta$, $V^I_t(G^*) \geq \epsilon/2$ as $\delta \to 0$, i.e. for small $\delta$, the incumbent’s value is strictly positive at all times. From Equation 13, we can see that for all $t \leq t^*$, $V^I_{ni,t}$ converges to zero as $\delta$ becomes small. Thus, given our assumptions, the incumbent’s participation constraints hold for all $t \leq t^*$ as $\delta$ becomes small.

We only have to deal with the case $t^* < t \leq t^* + \delta$, since there, the no-information value does not vanish.\(^{25}\) $V^I_t(G^*)$ converges to $pR / (pe^{-\lambda(t^*-t)} + 1 - p)$ as $\delta \to t^* - t$.\(^{26}\) The no-information value converges to $(1 - p)R / (pe^{-\lambda(t^*-t)} + 1 - p)$. Plugging in $p = 1/2 + \epsilon$, we thus have $V^I_t(G^*) > V^I_{ni,t}$ for sufficiently small $\delta$. We can show by a similar argument that the upstart’s participation constraints are be satisfied as well.

This shows that the schedule $G^*$ is feasible at all times for sufficiently small $\delta$. It remains to show that it is approximately optimal, i.e. no other feasible schedule can leave the principal with value higher than $\epsilon/c$. The argument is similar to the one we used in the proof of Proposition 6.

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\(^{25}\)This is also why moving the goalposts is only approximately optimal with two agents. We need to give the extra rents to guarantee that the incumbent’s participation constraint holds for $t > t^*$.

\(^{26}\)For $t > t^*$, $\delta$ can never be smaller than $t - t^*$, because otherwise $t$ would be larger than $t^* + \delta$, a contradiction.
For any schedule $G$, the incumbent’s value is

$$V^I(G) = \mathbb{E}_G \left[ 1_{x \leq t < x + \delta} e^{-rt} R - c (1 - e^{-rt}) \right] \leq \mathbb{E}_G [e^{-rx} R - c (1 - e^{-rt})] = V^I(G^*)$$

and the upstart’s value is

$$V^U(G) = \mathbb{E}_G \left[ 1_{t \geq x + \delta} e^{-rt} R - c (1 - e^{-rt}) \right] \leq \mathbb{E}_G [e^{-r(x+\delta)} R - c (1 - e^{-rt})] = V^U(G^*) .$$

If another schedule $G'$ satisfies

$$\mathbb{E}_{G'} [1 - e^{-rt}] \geq \mathbb{E}_{G^*} [1 - e^{-rt}] + \frac{\epsilon}{c},$$

this schedule violates the agent’s participation constraints, because their values under $G^*$ are below $\epsilon$. $G^*$ is therefore approximately optimal.  

$\square$