Inter-Dealer Trades in OTC Markets – Who Buys and Who Sells?

Chung-Yi Tse and Yujing Xu∗

February 19, 2018

Abstract

Oftentimes, dealers in an OTC market may not be able to trade with one another whenever they desire to do so, just as investors find it necessary to incur time and effort to buy and sell assets not traded in a centralized exchange. Moreover, an individual dealer can obviously only carry limited quantities of the asset over time and the inventory capacities may certainly differ among dealers. In this environment, dealers trade among themselves, whenever the opportunities arise, to rebalance inventories for facilitating the sale and purchase of the asset to and from investors. In equilibrium, the smaller-capacity dealers sell to the larger-capacity dealers when the asset supply is at a low level but buy from them when the asset supply is at a high level. If the smaller-capacity dealers are interpreted as the peripheral dealers and the larger-capacity dealers are interpreted as the core dealers in a core-periphery trading network, it is the peripheral dealers who trade to provide immediacy for the core dealers – a prediction, though counterintuitive, is supported by some available empirical evidence.

Keywords: OTC Market, Inter-Dealer Trades, Trading Network
JEL classifications: D53, D85, G23

∗Faculty of Business and Economics, University of Hong Kong. E-mail: tsechung@econ.hku.hk (Tse); yujingxu@hku.hk (Xu). We would like to thank Charles Leung, Dongkyu Chang, the seminar audiences of the City University of Hong Kong and the audiences of the 2017 Econometric Society Asian meeting for very helpful comments and suggestions. Xu gratefully acknowledges financial support from HK GRF grant 17517816.
1 Introduction

Many financial assets, including government and corporate bonds, asset-backed securities, and derivatives, are traded in over-the-counter (OTC) markets instead of in centralized exchanges.\(^1\) Two distinguishing features of OTC markets are that trades are almost always intermediated by dealers of various kinds and that the dealers do not just trade with investors but also among themselves. Indeed, inter-dealer trades can account for a significant fraction of the overall transactions for a given asset.\(^2\)

It has long been recognized, going back to Ho and Stoll (1983), that dealers may trade among themselves for inventory risk concerns. In these models, a risk-averse trader having a greater exposure to some risky assets sells a certain fraction of his holding to another risk-averse trader with a lesser initial exposure to the mutual benefits of both. The common understanding seems to be that trading for inventory concerns is inherently linked to risk aversion. But must inter-dealer trades motivated by the sharing of inventory risks necessarily arise from risk aversion?

In many OTC markets, a dealer cannot take up a certain buy order from an investor unless the dealer possesses a large enough inventory of the asset beforehand if the dealer is not able to acquire the requisite amount of it from other dealers or investors at a short notice. On the other hand, if dealers’ inventory capacities are not unbounded and if they cannot sell to others at will, a dealer can only meet a sell order from an investor if the dealer has sufficient spare inventory capacity at the given moment. Given such constraints, dealers may find it beneficial to trade with one another, whenever they are able to do so, to reach some optimal inventory levels to best prepare themselves for trading with investors. Such are inter-dealer trades in an OTC market motivated by the sharing of inventory risk, broadly understood, but not out of any consideration related to risk aversion.

In this paper, we extend the seminal random search models of the OTC market of Duffie, Gärleanu and Pedersen (2005) and Lagos and Rocheteau (2009) to study how dealers trade with one another for managing inventory levels for their future trading needs with investors. The point of departure is that, in our model, (1) dealers have only imperfect access to trading with other dealers and (2) they are heterogeneous in their inventory capacities. These are arguably very plausible assumptions. First, to be sure, in reality, there is not a frictionless platform on which dealers can continuously trade among themselves in a typical OTC market, just as investors must expend time and effort in buying and selling the asset. The heterogeneity in inventory capacity among dealers can result from differences in financing costs – dealers who finance asset purchases out of retained earnings and owners’ equities can face different opportunity costs of funds, whereas dealers who finance asset purchases by borrowing can be charged different risk premia. The heterogeneity can also be due to risk management considerations or portfolio choices.\(^3\)

\(^1\)As an example, the gross market value of global OTC derivatives totaled 38,286 billion US dollars in 2014 and 29,992 billion US dollars in 2015 (Bank for International Settlement, Semiannual OTC derivatives statistics, updated on May 4, 2016).

\(^2\)Li and Schürhoff (2014) show that in the period covered by their data set, 16 million out of 60 million transactions in municipal bonds are inter-dealer trades. A similar percentage of inter-dealer trade is also documented in Hollifield, Neklyudov and Spatt (2016).

\(^3\)In this paper, we do not attempt to model how the heterogeneity arises endogenously but instead restrict
With imperfect access to inter-dealer trading, it becomes imperative for dealers to choose the appropriate levels of inventory holding to be able to meet the uncertain future buy and sell orders from investors. In our model, dealers possessing different inventory capacities attain their respective optimal inventory holdings by buying and selling among themselves when the opportunities come, and that dealers of different inventory capacities play different roles in the inter-dealer market.

In particular, in our model, there is a given measure of what we call small dealers, each endowed with one unit of inventory capacity, and a given measure of what we call large dealers, each endowed with two units of inventory capacity. At the beginning of each period, investors who value the asset highly but have no asset in hand (high-valuation non-owners hereinafter) and investors who own a unit of the asset but do not value it (low-valuation owners hereinafter) enter the market to buy from and sell to dealers. Investors and dealers randomly meet in this investor-dealer market in which a given dealer can only sell to (buy from) an investor if the dealer has at least one unit of inventory (spare capacity) beforehand. Once the investor-dealer trades are completed, and only then, a perfectly competitive inter-dealer market opens, through which dealers can rebalance their inventory holdings.

Underlying most of the results in the paper is a particular ranking of the marginal benefits of inventory according to which both the first unit of inventory and the last unit of spare inventory capacity are valued higher by a large dealer than by a small dealer. The ranking is due to how the small, but not the large, dealer would exhaust his entire inventory capacity in acquiring one unit of the asset and how the large, but not the small, dealer need not fill up his entire inventory to already possess a unit of the asset for sale to investors.

The ranking implies that in equilibrium, all small dealers either sell or buy in the inter-dealer market. Large dealers, on the other hand, sell as well as buy in any equilibrium. This means that small dealers trade with large dealers only but not among themselves while trades between the two types of dealers flow in just one direction. The direction of trade between a given pair of small and large dealers is then persistent, a prediction consistent with the findings in Li and Schürhoff (2014).

Which direction the given trade takes turns out to depend on the asset supply in a rather surprising manner. If large dealers are able to and indeed tend to hold a larger inventory of the asset, perhaps it seems intuitive that they should sell to small dealers in equilibrium. We find that this is the case, however, only when the asset supply is relatively abundant, at which times small dealers should find it easiest to buy the asset themselves from investors. A dealer is said to provide immediacy to another dealer if the first dealer tends to sell to (buy from) the second dealer when it takes longest on average for the second dealer to buy the asset in the market as arising from, for example, a relatively meagre (abundant) asset supply. Apparently, the large dealers in our model do not provide immediacy for small dealers. Rather, it is the small dealers who provide immediacy for the large dealers by selling to the latter when the asset supply is at a relatively low level but buying from them when the asset supply is at a relatively high level.

It is well known that in many OTC markets, as documented in Li and Schürhoff (2014) and Hollifield, Neklyudov and Spatt (2016), a set of dealers, known as the core dealers, tend attention to exploring the consequences thereof.
to trade with all dealers in the market, whereas the rest, known as the peripheral dealers, only
trade with the core dealers. Given that all small dealers in our model only trade with the large
dealers, but not among themselves, whereas the large dealers trade with all dealers, the two
types of dealers behave similarly as the peripheral and core dealers do, respectively, identified
in the empirical studies, with regard to the set of dealers they are predicted to trade with.
Under this interpretation then, our model predicts that it is the small peripheral dealers who
provide immediacy for the large core dealers. The prediction seems counterintuitive. But there
indeed exists empirical evidence supporting it as we shall discuss in the following.

The equilibrium in our model is constrained efficient in that the allocation of inventories
and spare capacities among dealers falling out from inter-dealer trades in equilibrium coincide
with the planning optimum. We show that rather intuitively, the optimum allocations serve
to enable investors to buy and sell the asset most rapidly. The equilibrium allocations, on the
other hand, are such that inventories and spare capacities are held by dealers who value them
the most for their trading needs with investors. That the two allocations coincide perhaps is
not surprising but more interestingly, it suggests that for efficiency, small peripheral dealers
indeed should trade to provide immediacy for the large core dealers.

In addition to implications on the structure of trading relationships, our model yields a
rich set of other testable implications for future empirical research on the OTC market. We
investigate how market-tightness, inter-dealer trading price and volume change with respect to
the asset supply and the diversity of dealers. Perhaps somewhat unexpected a priori is that the
inter-dealer trading volume is “M-shaped” in response to changes in the asset supply – trading
is most active when the asset supply is at a moderately low, but not the lowest, level and at
a moderately high, but not the highest, level. Dealers trade among themselves to rebalance
inventory, to which the need is greatest when either they find it hardest to acquire inventory
or liquidity from investors, i.e., when the asset supply is at the lowest or the highest level.
But precisely when the asset supply is at the lowest or the highest level, dealers who possess
inventory (spare capacity) to sell (buy) can only be few and far between. In equilibrium,
prices must then rise (fall) to dampen the demand (supply). In this way, trading is most active
when the demand for and the supply of inventory are both at relatively high levels, arising from
there being a moderately high or low asset supply. The inter-dealer trading volume is also non-
monotonic with respect to the fraction of large dealers in the dealer population, reaching the
maximum level when the fraction is at some intermediate level, whereby the dealer population
is most diverse in terms of inventory capacity.

Related Literature

Our paper departs from early models of inter-dealer trades motivated by inventory risk
concerns that follow from Ho and Stoll (1983) and the modern variants, as in Atkeson, Eisefält
d and Weill (2015) and Üslü (2016), by assuming all traders are risk neutral. The dealers in our
model are trading to mitigate inventory risks, as those in these models do, in that they trade
to eliminate the risks of carrying an insufficient inventory or an insufficient inventory capacity
for their trading needs with investors.

Why dealers trade among themselves, other than for risk sharing, is a topic of active ongoing
research. Colliard and Demange (2017) study post-issuance intermediation chains where each
dealer has limited cash endowment, so that they need to trade with one another to amass
the cash endowment of a group of dealers. Glode and Opp (2016) argue that when there is an adverse selection problem, a longer intermediation chain can narrow the information gap between two successive dealers and help mitigate the problem of information asymmetry. Hugonnier, Lester and Weill (2016) and Shen, Wei and Yan (2016) assume that investors value the asset differently and show that those with intermediate valuations endogenously become dealers as they stay in the market to sell to investors with even higher valuations after buying.4

The primary contribution of our paper is an investigation of how the asset supply helps determine the direction of trades among different types of dealers and the implications thereof on who provide immediacy for whom in a simple and parsimonious model. A secondary contribution is a novel implication on how the volume of inter-dealer trades should vary with the asset supply non-monotonically. In other models of inter-dealer trades, it is not clear how a change in the asset supply may have any clear-cut implications on trading directions and how the asset supply-trading volume relation can be non-monotonic.

Our framework is adapted from the seminal models of OTC markets in Duffie, Gärleanu and Pedersen (2005) and Lagos and Rocheteau (2009), with two main differences being dealers’ imperfect access to inter-dealer trading and the heterogeneity of dealers’ inventory capacity – two features that make the present model more suitable for studying the inter-dealer trading relationship. In the two aforementioned papers, whenever a dealer trades with an investor, the dealer can instantaneously offset the transaction by trading in a perfectly competitive inter-dealer market that opens at all times. Such an environment, in which a dealer trades with another dealer only if and when he meets an investor, is arguably not the best environment to study inter-dealer trades as the trades have neither persistent direction nor particular structure.

A host of recent papers are motivated to explain the empirical finding that the inter-dealer market exhibits a core-periphery structure. For instance, Neklyudov (2015) proposes a random search model assuming that dealers differ in their search abilities. He shows that the dealers with higher search abilities are more interconnected and hence are in the center. We highlight another factor, inventory capacity, that can influence a dealer’s position in inter-dealer trading relationships.

Another strand of investigation, a notable example of which is Wang (2017), models explicitly the formation of a core-periphery trading network in an environment in which a dealer can only trade with another dealer with which it has previously paid to set up an account. These models study how most dealers would choose to set up accounts with just a handful of dealers due to the usual network externality. In our model, the trading structure that emerges endogenously resembles a core-periphery trading network without any kind of network externality and where the establishment of a prior link is not needed for any two dealers to trade.

Ours is not the only model of an OTC market in which dealers hold inventory. Lagos, Rocheteau and Weill (2011) demonstrate that dealers hold inventory to speed up future trades when there is a negative shock knocking the market off the steady state, even if dealers do not value the asset. Weill (2011) shows that the same intuition also applies in a competitive dynamic market with a transient selling pressure. Dealers in our model hold inventory also to facilitate trade, as they do not have continuous access to the inter-dealer market. The difference is that they hold inventory even in the steady state and they gain by trading among

---

4 A similar mechanism is proposed in Piazzesi and Schneider (2009) in their analysis of the housing market.
themselves.

For brevity, we restrict attention to studying steady-state equilibrium in this paper. The rest of the paper is organized as follows. In Section 2, we set up the model and then study the model’s equilibrium. We discuss the model’s implications on dealers’ markups that follow from its predictions on the directions of trade between small and large dealers in Section 3 and compare those implications against the available empirical evidence. In Section 4, we discuss three extensions of the model and demonstrate how the major results hold in more general settings. We return to our basic model in Section 5 to study the model’s comparative statics. Section 6 concludes with discussions on the constrained efficiency of equilibrium in particular. All proofs are relegated to the Appendix, which also includes three respective Sections for the details of one extension of the model, two Propositions on the comparative statics of prices, and the formal analysis of the constrained efficiency of equilibrium.

2 Model and Analysis

2.1 Basic Environment

Time is discrete and runs forever. Two groups of agents – investors and dealers – buy and sell an asset with supply fixed at $A$ in an OTC market. A high-valuation investor derives a per period return of $\nu > 0$ in holding a unit of the asset, whereas low-valuation investors and dealers derive the same per period return normalized to zero. An individual investor can hold either zero or one unit of the asset at a time and can only buy or sell the asset through dealers of which there are two types: (1) small dealers, each of whom can hold up to one unit of the asset at a time and (2) large dealers, each of whom can hold up to two units. All agents are risk neutral and discount the future at the same factor $\beta$.

At the beginning of each period, a measure of $e$ investors enter the market as high-valuation investors with no assets in hand. Together with the entrants in previous periods who have yet to acquire a unit of the asset, they – the high-valuation non-owners – constitute the population of investor-buyers in the market. Among investors who do own a unit of the asset, the low-valuation owners become the investor-sellers in equilibrium.

Each period is divided into two subperiods. In the first subperiod, a decentralized investor-dealer market opens in which the bilateral meetings between investors and dealers take place. We assume that investor-buyers and dealers with at least a unit of the asset for sale (dealer-sellers hereinafter) meet in one segment of the market and investor-sellers and dealers with spare inventory capacity to buy (dealer-buyers hereinafter) meet in another segment of the market. The matches in each market segment are formed in accordance with the same Mortensen-Pissarides constant-returns matching function, whereby, given market tightness $\theta \in [0, \infty)$ for the ratio of the measures of buyers to sellers for the given market segment, a seller meets a buyer at a probability $\eta(\theta) \in [0, 1]$, whereas a buyer meets a seller at the probability $\mu(\theta) = \eta(\theta) / \theta$. The meeting probability $\eta(\theta)$ satisfies the usual conditions:

$$\frac{\partial \eta}{\partial \theta} > 0; \quad \frac{\partial^2 \eta}{\partial \theta^2} < 0; \quad \lim_{\theta \to 0} \frac{\partial \eta}{\partial \theta} = 1; \quad \lim_{\theta \to \infty} \frac{\partial \eta}{\partial \theta} = 0.$$
With two market segments, there are two market tightness: (1) $\theta_{ID}$ for the ratio of the measures of investor-buyers to dealer-sellers and (2) $\theta_{DI}$ for the ratio of the measures of dealer-buyers to investor-sellers.

Prices in the investor-dealer market fall out of the bargaining between the buyers and sellers in the bilateral meetings in which the agents on the two sides are assumed to possess equal bargaining power.\(^5\) An individual dealer may search as both a dealer-buyer and a dealer-seller in the market in a given period but the meeting technology only allows the dealer to meet at most one investor-buyer and one investor-seller in the period.\(^6\) At the end of the subperiod, those high-valuation non-owners who succeed in buying a unit turn into new high-valuation owners, while those low-valuation owners who succeed in selling their units leave the market for good.

In the second subperiod, a competitive inter-dealer market opens, in which dealers buy and sell as many units of the asset among themselves as they see fit at a given market price, subject to their asset holdings and spare inventory capacities. Finally, at the end of the second subperiod, each high-valuation owner, except for those who have just purchased the asset in the current period, turns into a low-valuation owner at a probability $\delta \in (0, 1)$.

An apparent alternative to our assumed meeting technology is that investors and dealers search and match in one unified market in which an investor meets a randomly selected dealer at a probability that depends on the ratio of all dealers to all investors. In this setup, there would be bilateral meetings between two sellers and between two buyers that cannot lead to any profitable exchanges between the agents concerned. Such no-trade meetings should not be common occurrences in reality. Hendershott and Madhavan (2015) report the increasing prevalence of electronic trading platforms for corporate bonds on which investors post their buy and sell orders. A dealer in these markets is then usually well informed of whether an investor is buying or selling before he initiates contact with the investor. Our assumed meeting technology embodies the standard assumption that how many matches of one type are formed depends only on the measures of agents who may become partners in such matches, whereas in the unified market setup, the measures of other types of agents also matter and exert negative effects on the measures of the matches of the given type that would be formed, which means that for instance, an investor-buyer’s matching probability decreases with the measures of dealer-buyers and investor-sellers. Our assumed meeting technology has the virtue of choosing a simple versus a complicated setting when there is no compelling reason for choosing the latter.

In the models of Duflie, Gărleanu and Pedersen (2005) and Lagos and Rocheteau (2009), dealers do not hold any inventory at all in the steady state, as they can immediately offset any transaction with investors in an inter-dealer market that opens at all times. In contrast, the dealers in our model access the inter-dealer market only after or before they meet and trade with investors. Without continuous access to trading with other dealers, a dealer in our

\(^5\)The assumption of equal bargaining power is without loss of generality and merely serves to simplify.

\(^6\)This simplifying assumption can be understood as a discrete-time version of a continuous-time meeting process. If the time interval between two periods is small enough relative to the arrival rate of a meeting, then the probability of having more than one meeting per period approaches zero. The assumption can also be justified by a dealer’s limited execution capacity in reality. We should explain in Section 4 how the major results of the analysis survive while relaxing the restriction.
model can sell to an investor only if the dealer is holding at least a unit of the asset beforehand and thus the dealer may find it optimal not to offload all units of the asset he acquires from investors at the first opportunity. In a similar vein, a dealer may find it optimal not to entirely fill up his inventory in the inter-dealer market, in anticipation of using the spare capacity for trading with an investor-seller if and when he meets one in the next period.

2.2 Value Functions

A small dealer, \( S_i, i = 0, 1 \), is either holding 0 or 1 unit of the asset at the beginning of a period when the investor-dealer market opens, whereas a large dealer, \( L_i, i = 0, 1, 2 \), may also be holding up to 2 units of the asset.

In the investor-dealer market, an investor-buyer meets a dealer-seller at probability \( \mu (\theta_{ID}) \). The dealer-seller can be an \( S_1 \), an \( L_1 \) or an \( L_2 \). Let \( p_{IB,S_1}, p_{IB,L_1} \), and \( p_{IB,L_2} \) be the respective prices at which the investor-buyer buys from these different dealers. Then, the investor-buyer has asset value,

\[
U^B = \mu (\theta_{ID}) \left( \beta U^ON_H - \frac{n^SD_1}{n^SD} p_{IB,S_1} - \frac{n^LD_1}{n^SD} p_{IB,L_1} - \frac{n^LD_2}{n^SD} p_{IB,L_2} \right) + \left( 1 - \mu (\theta_{ID}) \right) \beta U^B, \tag{1}
\]

where \( n^SD_i \) and \( n^LD_i \) are the respective measures of small and large dealers holding an \( i \)-unit inventory,

\[
n^{D}_i = n^SD_i + n^{LD}_1 + n^{LD}_2
\]

the measure of all dealer-sellers, and \( U^ON_H \) the asset value of a high-valuation owner. In defining this value function and the ones that follow, we assume that all meetings in the investor-dealer market yield a non-negative match surplus. In Lemma 1 below, we show that the restriction is without loss of generality as it indeed holds in any equilibrium with active trading between investors and dealers.

A high-valuation owner derives a per period return \( \upsilon \) from holding a unit of the asset and may turn into a low-valuation owner at probability \( \delta \) at the end of the period. Hence,

\[
U_{ON}^H = \upsilon + \beta \left( \delta U_{ON}^L + \left( 1 - \delta \right) U_{ON}^H \right), \tag{2}
\]

where \( U_{ON}^L \) denotes the asset value of a low-valuation owner who seeks to sell his unit of the asset. In each period, the investor-seller meets a dealer-buyer at probability \( \eta (\theta_{DI}) \). The dealer may be an \( S_0 \), an \( L_0 \) or an \( L_1 \). Let \( p_{So,I_S}, p_{Lo,I_S}, \) and \( p_{L_1,I_S} \) be the respective prices at which the low-valuation investor sells to these different dealers. Hence,

\[
U_{ON}^L = \eta (\theta_{DI}) \left( \frac{n^SD_0}{n_B} p_{So,I_S} + \frac{n^LD_0}{n_B} p_{Lo,I_S} + \frac{n^{LD}_1}{n_B} p_{L_1,I_S} \right) + \left( 1 - \eta (\theta_{DI}) \right) \beta U_{ON}^L, \tag{3}
\]

where

\[
n^{D}_B = n^SD_0 + n^{LD}_0 + n^{LD}_1
\]

is the measure of all dealer-buyers.\footnote{In holding a unit in inventory and having one unit of spare inventory capacity, an \( L_1 \) is both a dealer-seller and a dealer-buyer in the given period.}
In addition to trading with investors in the investor-dealer market in the first subperiod, dealers may also trade among themselves in the second subperiod in the competitive inter-dealer market. Write $V_i^{SD}$ and $W_i^{SD}$, $i = 0, 1$, as the respective asset values of a small dealer entering the investor-dealer market in the first subperiod and the inter-dealer market in the second subperiod with an $i$-unit inventory. If the asset is traded in the inter-dealer market at price $p$,

$$W_0^{SD} = \max \left\{ \beta V_0^{SD}, \beta V_1^{SD} - p \right\},$$ (4)

$$W_1^{SD} = \max \left\{ p + \beta V_0^{SD}, \beta V_1^{SD} \right\},$$ (5)

$$V_0^{SD} = \mu (\theta_{DI}) (W_1^{SD} - p_{S_0,I_2}) + (1 - \mu (\theta_{DI})) W_0^{SD},$$ (6)

$$V_1^{SD} = \eta (\theta_{ID}) (p_{I_2,S_1} + W_0^{SD}) + (1 - \eta (\theta_{ID})) W_1^{SD}.$$ (7)

In (4), an $S_0$ entering the inter-dealer market chooses between buying a unit in the market and not buying, whereas in (5), an $S_1$ chooses between selling the unit and not selling. Clearly, if the first dealer strictly prefers to buy where $p < \beta (V_1^{SD} - V_0^{SD})$, the second dealer must strictly prefer not to sell and vice versa. In (6), an $S_0$ entering the investor-dealer market meets an investor-seller at probability $\mu (\theta_{DI})$ and buys the unit from the investor at price $p_{S_0,I_2}$. In (7), an $S_1$ meets an investor-buyer at probability $\eta (\theta_{ID})$ and sells the unit to the investor at price $p_{I_2,S_1}$.

A large dealer can hold up to two units of the asset in inventory. The asset values, $V_i^{LD}$ and $W_i^{LD}$, $i = 0, 1, 2$, satisfy, respectively,

$$W_0^{LD} = \max \left\{ \beta V_0^{LD}, \beta V_1^{LD} - p, \beta V_2^{LD} - 2p \right\},$$ (8)

$$W_1^{LD} = \max \left\{ p + \beta V_0^{LD}, \beta V_1^{LD}, \beta V_2^{LD} - p \right\},$$ (9)

$$W_2^{LD} = \max \left\{ 2p + \beta V_0^{LD}, p + \beta V_1^{LD}, \beta V_2^{LD} \right\},$$ (10)

$$V_0^{LD} = \mu (\theta_{DI}) (W_1^{LD} - p_{L_0,I_2}) + (1 - \mu (\theta_{DI})) W_0^{LD},$$ (11)

$$V_1^{LD} = \mu (\theta_{DI}) (1 - \eta (\theta_{ID})) (W_2^{LD} - p_{L_1,I_2}) + (1 - \mu (\theta_{DI})) (p_{I_2,L_1} + W_0^{LD}) + \mu (\theta_{DI}) \eta (\theta_{ID}) (p_{I_2,L_1} - p_{L_1,I_2} + W_1^{LD}) + (1 - \mu (\theta_{DI})) (1 - \eta (\theta_{ID})) W_1^{LD},$$ (12)

$$V_2^{LD} = \eta (\theta_{ID}) (p_{I_2,L_2} + W_1^{LD}) + (1 - \eta (\theta_{ID})) W_2^{LD}.$$ (13)

The prices for the investor-dealer trades in the above are assumed to be determined by standard Nash bargaining in which the two agents of a given trade possess equal bargaining power.8

8See (52)-(57) in the proof of Lemma 1 in the Appendix for the pricing equations.
2.3 Prices and Match Surpluses in the Investor-dealer Market

As we remarked earlier, in defining the value functions in (1)-(3), (6) and (7), and (11)-(13), we assume that there are non-negative match surpluses in any and all meetings in the investor-dealer market. A priori this need not be true as it is not inconceivable that a given type of dealer holding a particular inventory may only find it profitable to trade with other dealers but not with investors. By Lemma 1 below, the restriction is without loss of generality in any steady-state equilibrium with active trading.

Lemma 1 In any steady-state equilibrium, the match surplus for meetings between an investor-seller and any dealer-buyer all equals to

\[ z_{IS} = p - \beta U_{ON}^L, \]  

while any such exchanges take place at the same price,

\[ p_{S0,IS} = p_{L0,IS} = p_{L1,IS} = \frac{p + \beta U_{ON}^L}{2}. \]  

The match surplus for meetings between an investor-buyer and any dealer-seller all equals to

\[ z_{IB} = \beta (U_{ON}^H - U^B) - p, \]  

while any such exchanges take place at the same price,

\[ p_{IB,S1} = p_{IB,L1} = p_{IB,L2} = \frac{p + \beta (U_{ON}^H - U^B)}{2}. \]  

In a narrow sense, the Lemma holds because of a competitive inter-dealer market in which dealers buy and sell the asset at the same price \( p \). In this case, if a dealer buying a unit from an investor finds it optimal to sell it in the inter-dealer market right after, the dealer earns a surplus equal to the selling price \( p \) minus the payment he makes to the investor, whereas if the dealer finds it optimal to keep the unit for selling to other investors later on, he earns just the same surplus as the unit would have cost him the buying price \( p \) in the inter-dealer market had he not bought it earlier from the investor. On the other hand, to the investor-seller, the gain from trade is equal to the payment from the dealer net of the continuation value of being a low-valuation owner, which is otherwise independent of the identity of the counterparty of the trade. The match surplus, equal to the sum of the surpluses from trade of the two sides, is then the same across all trades between an investor-seller and any dealer-buyer as given in (14). A similar logic explains (16). In a broader sense, as we shall show in Lemma 4 when we extend the analysis to a frictional inter-dealer market, that any trade between a dealer and an investor-seller should yield a non-negative surplus is that had the dealer found it not optimal to buy from an investor, the dealer could only be buying the asset from another dealer, who had bought a unit from an investor earlier, for him to be an active agent in equilibrium. But then there cannot be any greater surplus for a unit to be acquired by one dealer first and then passed onto another dealer than for the unit to be bought by the latter dealer in the first instance.
2.4 Inter-dealer Market Trades

By (4) and (5), whether a small dealer entering the inter-dealer market wants to buy or sell depends on how the inter-dealer market price \( p \) compares with \( \beta (V_{1}^{SD} - V_{0}^{SD}) \). Similarly, by (8)-(10), a large dealer entering the market decides to buy or sell by comparing \( p \) against \( \beta (V_{1}^{LD} - V_{0}^{LD}) \) and \( \beta (V_{2}^{LD} - V_{1}^{LD}) \). To proceed, we first establish that:

**Proposition 1** \( V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD} \geq V_{2}^{LD} - V_{1}^{LD} \) in any active steady-state equilibrium in which \( z_{Is} \geq 0 \) and \( z_{Ib} \geq 0 \). The first inequality is strict if \( z_{Is} > 0 \), whereas the second inequality is strict if \( z_{Ib} > 0 \).

Proposition 1 says that in an active steady-state equilibrium, an \( L_{0} \) entering the inter-dealer market has the most to gain from acquiring a unit of the asset in the market, followed by an \( S_{0} \), whereas an \( L_{1} \) has to the least to gain. Intuitively, \( V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD} \) because the opportunity cost for the large dealer in utilizing his first unit of inventory capacity should be lower than the opportunity cost for the small dealer in utilizing his only unit of inventory capacity – in acquiring a unit in the inter-dealer market, the large dealer, but not the small dealer, still has spare inventory capacity to buy one more unit from an investor in the next period to capture any possible surplus of trade. If the latter surplus is strictly positive (\( z_{Is} > 0 \)), then a large dealer gains strictly more from the first unit of inventory than a small dealer does. When acquiring a unit in the inter-dealer market is at the expense of exhausting one’s inventory capacity for both the large and small dealers, however, the small dealer should have more to gain than the large dealer (\( V_{1}^{SD} - V_{0}^{SD} \geq V_{2}^{LD} - V_{1}^{LD} \)) since the large dealer holding a one-unit inventory already has a unit for sale to investors in the upcoming period, whereas the small dealer does not. If there is a strictly positive surplus in an investor-buyer-dealer-seller trade (\( z_{Ib} > 0 \)), the small dealer gains strictly more from the last unit of inventory than the large dealer does.

The ranking in Proposition 1 implies that \( L_{0}S \) at least weakly prefer to buy and \( L_{0}S \) at least weakly prefer to sell in equilibrium. Who else will buy and sell depends on what price clears the inter-dealer market, a price that must be bounded by

\[
p \in [\beta (V_{2}^{LD} - V_{1}^{LD}), \beta (V_{1}^{LD} - V_{0}^{LD})],
\]

since at any \( p \) above the upper bound of the interval, there can only be sellers and at any \( p \) below the lower bound, there can only be buyers in the market. Besides, for any \( p \) not exactly equal to \( \beta \) times one of the three marginal benefits in Proposition 1, any and all dealers who desire to trade either strictly prefer to buy or sell. In this case, the market clears only if the parameters conspire to just equate the measures of buyers and sellers. Such a parameter configuration, however, can only make up a zero-measure subset of the parameter space. Equilibrium obtains in general only for \( p \) just equal to \( \beta (V_{1}^{LD} - V_{0}^{LD}) \), \( \beta (V_{1}^{SD} - V_{0}^{SD}) \), or \( \beta (V_{2}^{LD} - V_{1}^{LD}) \), at which there is one type of dealer holding a given inventory indifferent between selling and not selling or between buying and not buying. The market may then clear at some particular mixing probability for the mixed strategy played by the marginal buyers or sellers. Furthermore, we can show that:

**Lemma 2** For \( p = \beta (V_{1}^{LD} - V_{0}^{LD}) \) or \( \beta (V_{1}^{SD} - V_{0}^{SD}) \), both \( z_{Is} \) and \( z_{Ib} \), and \( p \) itself are strictly positive, whereas for \( p = \beta (V_{2}^{LD} - V_{1}^{LD}) \), \( z_{Is} \) and \( p \) itself are equal to zero while
In all cases, the candidate equilibria are active equilibria in which the gains from trade between investors and dealers are non-negative.

The case for \( p = \beta \left( V_2^{LD} - V_1^{LD} \right) \) deserves further explanations. At this \( p \), an \( L_1 \) entering the inter-dealer market feels indifferent between paying \( p \) to buy one more unit and not buying. If he does buy, he exhausts his entire inventory capacity and the purchase would be at the expense of giving up the opportunity to buy from an investor in the next period. Meanwhile, given that he already possesses a unit in inventory to begin with, he can sell to an investor in the next period without buying a unit in the inter-dealer market. Thus, the dealer must be worse off acquiring a unit at any positive \( p \). With \( p = 0 \), a dealer is willing to buy a unit from an investor also only at a zero price, which means that there must be but a zero surplus in an investor-seller-dealer-buyer trade \( (z_{Is} = 0) \). On the other hand, there must be a positive surplus in an investor-buyer-dealer-seller trade \( (z_{Ib} > 0) \) since the investor, but not the dealer, is strictly better off owning a unit than not owning.

It is useful to classify equilibrium into three types, corresponding to \( p \) equal to each candidate equilibrium price.

**The “Selling” Equilibrium** In the Selling Equilibrium, \( p = \beta \left( V_1^{LD} - V_0^{LD} \right) \). By Proposition 1 and Lemma 2,

\[
p = \beta \left( V_1^{LD} - V_0^{LD} \right) > \beta \left( V_1^{SD} - V_0^{SD} \right) > \beta \left( V_2^{LD} - V_1^{LD} \right),
\]

in which case no dealers strictly prefer to buy, whereas any dealers, large and small, with a filled inventory strictly prefer to sell. For this reason, we call this the Selling Equilibrium in which the optimal inventory level of a large dealer is zero or one unit, whereas that of a small dealer is zero unit. Even though \( L_0 \)s are indifferent between buying and not buying, at least a fraction of them must buy in equilibrium since they are the only possible buyers. The market clears with a particular fraction of \( L_0 \)s each buying a unit and possibly a certain fraction of \( L_1 \)s each selling a unit when the net volume of purchase among the \( L_0 \)s and \( L_1 \)s is equal to the supply out of all \( S_1 \)s and \( L_2 \)s choosing to sell a unit each. Given that any equilibrium trading pattern should facilitate all dealers reaching their respective optimal inventories, the equilibrium with the least transactions, in which all \( L_1 \)s refrain from selling so that all purchases made by \( L_0 \)s are for meeting the supply from the inframarginal sellers, is arguably the most compelling one, among the continuum. Indeed, that equilibrium is the unique equilibrium if there is a vanishingly small, but positive, cost for each sale/purchase to be executed – an assumption we should maintain in the following – whereby, with no gains from trade, no \( L_1 \)s will incur the trading cost to sell.\(^{11}\)

\(^9\)A \( p = 0 \) results from the normalization that both low-valuation owners and dealers derive zero flow payoff from owning the asset. At a positive normalized payoff, \( p \) would become positive without affecting the qualitative results to follow.

\(^{10}\)In case investors can trade among themselves, an investor-seller should always be able to sell to an investor-buyer at a positive price, given that the latter gains from holding a unit of the asset. In the present setting, the investor-seller does not have direct access to trading with an investor-buyer but can only trade with a dealer, who may possibly not gain from acquiring the unit in which case the trade can only take place at a zero price.

\(^{11}\)No \( L_0 \)s will be willing to trade too if they are obliged to pay the transaction cost. The \( L_2 \)s and \( S_1 \)s, however, would be willing to finance the transactions with the \( L_0 \)s since they strictly prefer to trade.
Let $m^i_{SD}, i = 0, 1$ and $m^i_{LD}, i = 0, 1, 2$, be the respectively measures of small and large dealers entering the inter-dealer market holding an $i$-unit inventory. Then, the market clears if there are sufficiently many marginal buyers to meet the supply from the inframarginal sellers; i.e.,

$$m^1_{SD} + m^2_{LD} \leq m^0_{LD}. \quad (18)$$

The “Balanced” Equilibrium In the Balanced Equilibrium, $p = \beta (V^1_{SD} - V^0_{SD})$. By Proposition 1 and Lemma 2,

$$\beta (V^1_{LD} - V^0_{LD}) > p = \beta (V^1_{SD} - V^0_{SD}) > \beta (V^2_{LD} - V^1_{LD}),$$

from which it follows that $L_0$s strictly prefer to buy one unit while $L_2$s strictly prefer to sell one unit. We refer to this as the Balanced Equilibrium, in which the optimal inventory level of a large dealer is one unit, whereas that of a small dealer is zero or one unit. For the inter-dealer market to clear, if large dealers selling (buying) outnumber large dealers buying (selling) in the market, the net demand (supply) by small dealers must just suffice to meet the net supply (demand) from large dealers. With a small but positive cost of trade, a fraction of $S_0$s buy, while no $S_1$s sell in case the net supply from large dealers is positive ($m^2_{LD} \geq m^0_{LD}$). Market clearing requires

$$m^1_{LD} - m^0_{LD} \leq m^0_{SD}. \quad (19)$$

Otherwise ($m^2_{LD} \leq m^0_{LD}$), a fraction of $S_1$s sell, while no $S_0$s buy in the unique equilibrium. The inter-dealer market clears if

$$m^0_{LD} - m^2_{LD} \leq m^1_{SD}. \quad (20)$$

The “Buying” Equilibrium In the Buying Equilibrium, $p = \beta (V^2_{LD} - V^1_{LD})$. By Proposition 1 and Lemma 2,

$$\beta (V^1_{LD} - V^0_{LD}) = \beta (V^1_{SD} - V^0_{SD}) > \beta (V^2_{LD} - V^1_{LD}) = p = 0,$$

from which it follows that no dealers strictly prefer to sell. Meanwhile, any dealers with an empty inventory, large and small, strictly prefer to buy. For this reason, we call this the Buying Equilibrium in which the optimal inventory level of a large dealer is one or two units, whereas that of a small dealer is one unit. For the inter-dealer market to clear, a fraction of $L_2$s sell since they are the only possible sellers, while all $L_1$s refrain from buying if trading is costly. For the inter-dealer market to clear,

$$m^0_{SD} + m^2_{LD} \leq m^2_{LD}. \quad (21)$$

Gains from trade in our model arise (1) out of dealers not having continuous access to trading among themselves and (2) from dealers having different inventory capacities. The former incentivizes dealers to manage inventories and the latter generates different optimal inventories among dealers. Because $L_0$s value a unit of inventory the most while $L_2$s the least, the equilibrium price must settle at where they gain from trading with one another.
There can also be mutually beneficial trades between a pair of large and small dealers when one dealer’s inventory falls short of while the other dealer’s inventory exceeds their respective optimal inventories. On the other hand, at a price such that it is optimal for an $S_1$ to sell where the receipt of the selling price $p$ more than offsets the lost of the unit inventory for the small dealer, it cannot be optimal for an $S_0$ to buy since in this case, the gain from acquiring the unit inventory cannot cover the purchase price $p$. All this means that in equilibrium, either all small dealers sell or all small dealers buy. Small dealers then cannot be observed to be trading among themselves.

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Price</th>
<th>Optimum Inventory</th>
<th>Buyers</th>
<th>Sellers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>small dealers</td>
<td>large dealers</td>
<td></td>
</tr>
<tr>
<td>Selling</td>
<td>$p = \beta (V_1^{LD} - V_0^{LD})$</td>
<td>0</td>
<td>1 and 0</td>
<td>$L_0$</td>
</tr>
<tr>
<td>Balanced</td>
<td>$p = \beta (V_1^{SD} - V_0^{SD})$</td>
<td>0 and 1</td>
<td>1</td>
<td>$L_0$, $S_1$</td>
</tr>
<tr>
<td>Buying</td>
<td>$p = \beta (V_2^{LD} - V_1^{LD})$</td>
<td>1</td>
<td>1 and 2</td>
<td>$S_0$, $L_0$</td>
</tr>
</tbody>
</table>

Table 1: Prices, Optimal Inventories, Buyers and Sellers in Equilibrium

In Table 1, we summarize the optimal inventories of large and small dealers upon exiting and the identities of the buyers and sellers upon entry into the inter-dealer market, where the second line of each cell of the “Buyers” and “Sellers” columns indicate the identities of the marginal buyers and sellers in the three types of equilibrium. The Table shows that in any equilibrium, $L_1$s never trade, at least a fraction of $L_0$s buy and at least a fraction of $L_2$s sell. What differs among the equilibria is the role played by small dealers. In the Selling Equilibrium, $S_1$s sell while $S_0$s stay out of the market. In the Buying Equilibrium, $S_0$s buy while $S_1$s stay out of the market. In the Balanced Equilibrium, small dealers may either sell or buy, depending on whether or not the buyers among large dealers outnumber the sellers.

A direct implication of the results in Table 1 is that while there is no persistent trading direction among large dealers since they sell as well as buy in any equilibrium, small dealers either always sell to or buy from large dealers in a given type of equilibrium. Now, if only one type of equilibrium can hold for a given parameter configuration – a result we will establish in Proposition 2 to follow – the direction of trade between small and large dealers is persistent. The implication is consistent with the findings in Li and Schürhoff (2014) that given that there is a directional (buy or sell) trade between two dealers in one month, the probability that the same directional trade remains in the next month is 62%.

Our major results thus far – Proposition 1 and the implications thereof – seemingly rest on a number of simplifying assumptions. First, if small dealers can only hold at most one unit of the asset in inventory, it may seem trivial that they never trade among themselves in equilibrium. An obvious question to ask is how the results may hold if small dealers do gain by trading among one another in case they each possess more than a unit inventory capacity. Second, an important lesson in Li and Schürhoff (2014) and the follow-up study in Henderschott, Li, Livdan and Schürhoff (2016) is that the inter-dealer market is itself a
decentralized market as opposed to a Walrasian market. Finally, if large dealers can hold up to two units in inventory and may possess up to two units of spare inventory capacity, perhaps a more natural assumption is that they can meet up to two investor-buyers and two investor-sellers in each period. We will discuss how Proposition 1 and its main implications survive all three generalizations in Section 4 below.

A given type of equilibrium places a set of restrictions on the measures of dealer-buyers, dealer-sellers and the asset held by these dealers, which in turn impact on the probabilities at which investors and dealers buy and sell in the investor-dealer market. Then, a candidate equilibrium can indeed be equilibrium only if these restrictions are met and where the probabilities of trades are bounded below one, in addition to the existence of some positive mixing probability for the marginal buyers’ or sellers’ mixed strategy which clears the inter-dealer market. We now proceed to study the underlying environment as defined by the asset supply $A$, the turnover rate of high-valuation owners $\delta$, the entry rate of high-valuation non-owners $e$, and the measures of large and small dealers in which the restrictions of each type of equilibrium are met.

### 2.5 Accounting Identities, Market Tightness, and Stock-Flow Equations

If the market is populated by $n_{SD}$ small dealers and $n_{LD}$ large dealers,

\begin{align}
  n_{0}^{SD} + n_{1}^{SD} &= n_{SD}, \\
  n_{0}^{LD} + n_{1}^{LD} + n_{2}^{LD} &= n_{LD}.
\end{align}

The asset is in fixed supply equal to $A$, and hence,

\begin{align}
  n_{ON}^{H} + n_{ON}^{L} + n_{SD}^{1} + n_{LD}^{1} + 2n_{2}^{LD} &= A,
\end{align}

where $n_{ON}^{H}$ and $n_{ON}^{L}$ denote, respectively, the measures of high-valuation owners and low-valuation owners.

Let $n_{IB}^{I}$ denote the measure of high-valuation non-owners *cum* investor-buyers. Then,

\begin{align}
  \theta_{ID} &= \frac{n_{IB}^{I}}{n_{SD}^{I}} = \frac{n_{IB}^{I}}{n_{SD}^{I} + n_{LD}^{I} + n_{1}^{LD}}.
\end{align}

Recall that the population of investor-sellers is comprised of the low-valuation owners. Then,

\begin{align}
  \theta_{DI} &= \frac{n_{IB}^{I}}{n_{LD}^{I}} = \frac{n_{SD}^{I} + n_{LD}^{I} + n_{1}^{LD}}{n_{ON}^{L}^{I}}.
\end{align}

In the steady state, the respective inflows and outflows of high-valuation owners, low-valuation owners, and investor-buyers are equal. Hence,

\begin{align}
  n_{IB}^{I} \mu (\theta_{ID}) &= \delta n_{ON}^{H}, \\
  \delta n_{ON}^{H} &= \eta (\theta_{DI}) n_{ON}^{L}.
\end{align}

15
\[ e = n_B^I \mu (\theta_{ID}) . \]

Not all \( n_{SD}^i \) and \( n_{LD}^i \) can be positive in a given type of equilibrium. In the Selling Equilibrium for example, by the third and fourth columns of Table 1, all small dealers exit the inter-dealer market with an empty inventory, whereas large dealers may do so with either an empty or a one-unit inventory. The restrictions on the measures of small and large dealers with various levels of inventory in the three types of equilibrium are as depicted in Table 2.

<table>
<thead>
<tr>
<th></th>
<th>Selling Equilibrium</th>
<th>Balanced Equilibrium</th>
<th>Buying Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_0^{SD} )</td>
<td>( n^{SD} )</td>
<td>( 0, n^{SD} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( n_1^{SD} )</td>
<td>( 0 )</td>
<td>( 0, n^{SD} )</td>
<td>( n^{SD} )</td>
</tr>
<tr>
<td>( n_0^{LD} )</td>
<td>( 0, n^{LD} )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( n_1^{LD} )</td>
<td>( 0, n^{LD} )</td>
<td>( n^{LD} )</td>
<td>( 0, n^{LD} )</td>
</tr>
<tr>
<td>( n_2^{LD} )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0, n^{LD} )</td>
</tr>
</tbody>
</table>

Table 2: Measures of dealers entering the investor-dealer market in equilibrium

Given the measures of dealers, \( n_{SD}^i, i = 0, 1, \) and \( n_{LD}^i, i = 0, 1, 2, \) when the investor-dealer market opens in the first subperiod, the corresponding measures of dealers leaving the market and entering the inter-dealer market in the second subperiod are given by the following.

\[
m_0^{SD} = (1 - \mu (\theta_{DI})) n_0^{SD} + \eta (\theta_{ID}) n_1^{SD}, \tag{30}
\]

\[
m_1^{SD} = \mu (\theta_{DI}) n_0^{SD} + (1 - \eta (\theta_{ID})) n_1^{SD}, \tag{31}
\]

\[
m_0^{LD} = (1 - \mu (\theta_{DI})) (n_0^{LD} + \eta (\theta_{ID}) n_1^{LD}), \tag{32}
\]

\[
m_1^{LD} = \mu (\theta_{DI}) n_0^{LD} + [\mu (\theta_{DI}) \eta (\theta_{ID}) + (1 - \mu (\theta_{DI})) (1 - \eta (\theta_{ID}))] n_1^{LD} + \eta (\theta_{ID}) n_2^{LD}, \tag{33}
\]

\[
m_2^{LD} = (1 - \eta (\theta_{ID})) (\mu (\theta_{DI}) n_1^{LD} + n_2^{LD}). \tag{34}
\]

For example, (30) says that \( S_0 \)'s entering the inter-dealer market are among the \( S_0 \)'s entering the investor-dealer market who fail to buy a unit in the market and the \( S_1 \)'s who succeed in selling the unit they each possess.

### 2.6 Equilibrium

Given \( \{ n_{SD}^i, n_{LD}^i, A, e, \delta \} \), a steady-state equilibrium consists of the respective non-negative values of \( n_0^{SD}, n_1^{SD}, n_0^{LD}, n_1^{LD}, n_2^{LD}, n_H^{ON}, n_L^{ON} \) and \( n_B^I \) that satisfy (22)-(29), the restrictions on \( n_{SD}^i \) and \( n_{LD}^i \) in Table 2 and the market-clearing conditions for the type of equilibrium under consideration in (18)-(21), with \( m_i^{SD} \) and \( m_i^{LD} \) given by (30)-(34). Write \( n^D = n_{SD}^i + n_{LD}^i \) as the total measure of dealers.

**Proposition 2** The Selling Equilibrium and the Buying Equilibrium may only hold for \( e < n_{LD}^i \). The Balanced Equilibrium may only hold for \( e < n_{LD}^i + \frac{n_{SD}^i}{2} \).
a. For $e < n^{LD}$, define

$$B_S \equiv e + \frac{n^D}{\mu^{-1}\left(\frac{\mu}{n}\right)}$$

$$B_M \equiv n^{LD} + \frac{n^D}{\mu^{-1}\left(\frac{\mu}{n}\right)}$$

$$B_L \equiv n^D + \frac{n^{LD}}{\mu^{-1}\left(\frac{\mu}{n}\right)}$$

where $B_S \leq B_M \leq B_L$.

(i) for $A - e/\delta \in (B_S, B_M]$, the Selling Equilibrium holds,
(ii) for $A - e/\delta \in [B_M, B_L]$, the Balanced Equilibrium holds,
(iii) for $A - e/\delta \geq B_L$, the Buying Equilibrium holds.

b. For $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$, the Balanced Equilibrium exists if

$$A - \frac{e}{\delta} > e + \frac{n^D + n^{LD} - e}{\mu^{-1}\left(\frac{\mu}{n^D + n^{LD} - e}\right)} \equiv B_M.$$ 

c. In the Balanced Equilibrium,

(i) for $A - e/\delta < S$, small dealers sell in equilibrium,
(ii) for $A - e/\delta > S$, small dealers buy in equilibrium,
(iii) for $A - e/\delta = S$, small dealers do not trade in the inter-dealer market, where

$$S \equiv n^{LD} + \frac{n^{SD}}{2} + \frac{n^{LD} + \frac{n^{SD}}{2}}{\mu^{-1}\left(\frac{\mu}{n^{LD} + \frac{n^{SD}}{2}}\right)}.$$ 

For $e < n^{LD}$, $B_M \leq S \leq B_L$, whereas for $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$, $B_M \leq S$.

First, a steady-state equilibrium exists only if the necessary conditions on $e$, the entry rate of investors, stated at the beginning of the Proposition, are met. These conditions arise because if a measure of $e$ investors enter the market as high-valuation non-owners in each period, in the steady state, there have to be the same measure of $e$ investors exiting high-valuation non-ownership after buying a unit and the same measure of $e$ investors exiting low-valuation ownership and the market altogether after selling their units in the same period. All this requires that more than $e$ dealer-sellers and more than $e$ dealer-buyers are present in the market. In the Selling Equilibrium, only large dealers are carrying inventories for sale, whereas in the Buying Equilibrium, only large dealers possess spare capacities to buy. Then, for either type of equilibrium to exist, a necessary condition is that $e < n^{LD}$. In the Balanced Equilibrium, however, a fraction of small dealers enter the investor-dealer market with an
empty inventory and a fraction enter holding a one-unit inventory. Then, there will also be small dealers among both dealer-buyers and dealer-sellers and a steady-state equilibrium may exist even for $e \geq n^{LD}$ but not for $e \geq n^{LD} + \frac{n^{SD}}{2}$ since if more than one-half of all small dealers search as buyers (sellers), then there can only be fewer than one-half searching as sellers (buyers). In Figure 1, we illustrate how a steady-state equilibrium may only exist for $e < n^{LD} + \frac{n^{SD}}{2}$ and that for $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right]$, the only type of equilibrium that can hold is the Balanced Equilibrium.

The conditions in Parts (a)-(c) of the Proposition can be interpreted as conditions on the inventory of the asset held by dealers,

$$A^D \equiv A - n^O_H - n^O_L = A - \frac{e}{\delta} - \frac{n^D_B}{\mu - 1 \left(\frac{e}{n^O_H}\right)},$$

for each type of equilibrium to hold.\textsuperscript{12} In a given type of equilibrium, $A^D$ is first of all bounded by the measures of dealers who may be holding inventory of the asset. For example, in the Selling Equilibrium, small dealers do not hold any inventory while a fraction of large dealers may each hold a unit, in which case

$$0 \leq A^D \leq n^{LD}.$$

\textsuperscript{12}In each period in the steady state, a measure of $e$ investors enter high-valuation ownership, whereas each exits at the rate $\delta$, from which $n^O_H = e/\delta$ follows. Meanwhile, with the measure of low-valuation owners selling in each period equal to $e$, $\eta(\theta_D^T)n^O_L = e$, from which $n^O_L = n^T_B/\mu - 1 \left(\frac{e}{n^O_H}\right)$ obtains given $\theta_D^T = n^D_B/n^O_L$. 

Figure 1: Equilibrium – Existence and Uniqueness
Moreover, in the steady state, on the one hand, dealers’ asset holding must be sufficiently plentiful for them to sell $e$ units to the high-valuation non-owners. On the other hand, it must not exceed the level that would leave the dealers with insufficient spare inventory capacity to buy $e$ units from the low-valuation owners. All together, $A^D$ must be bounded by

$$e < A^D < n^{LD} + n^D - e.$$

(35)

It can then be shown that combining the two requirements yield the conditions in Parts (a)-(c).

Among the three types of equilibrium, the Selling Equilibrium, in which only a fraction of large dealers may hold just a one-unit inventory, involves dealers holding the least inventory, whereas the Buying Equilibrium, in which only a fraction of large dealers may still have one unit of spare inventory capacity, involves dealers holding the largest inventory. Part a(i) of the Proposition can be interpreted to say that when $A^D$, held entirely by large dealers, just suffices to satisfy the demand from investor-buyers, the Selling Equilibrium begins to hold and it holds until $A^D$ is up to the level at which all large dealers are holding a unit. At this point, according to Part a(ii), the Balanced Equilibrium begins to hold and it holds until $A^D$ is up to the level at which all small dealers are holding a unit in inventory as well. Thereafter, by Part a(iii), the Buying Equilibrium holds, in which all dealers hold at least a one-unit inventory and a fraction of large dealers are holding a two-unit inventory. Part (b) of the Proposition, as in Part (a), says that a steady-state equilibrium exists once $A^D$ is up to the level to satisfy investors’ asset demand. Figure 1, with $e$ on the horizontal axis measuring investors’ asset demand and $A$ on the vertical axis measuring the asset supply, where a larger $A$ corresponds to a larger $A^D$, illustrates the partition of $e$-$A$ space for the three types of equilibrium to hold.

Parts a(iii) and (b) of the Proposition say that a steady-state equilibrium holds even for arbitrarily large $A - e/\delta$, which can be interpreted as how the upper bound on $A^D$ in (35) will never be reached. In our model, as the asset supply increases and the sellers’ side of the investor-dealer market is becoming increasingly congested, it takes longer and longer for each low-valuation owner to sell, during which the measure of low-valuation owners and their asset holdings increase without bounds. With $n^{DN}_L$ increasing in tandem with the asset supply, the inventory held by dealers never rises above the level that would leave them with insufficient spare inventory capacity to buy $e$ units of the asset from investors.

In Part (c) of the Proposition, when $A - e/\delta = S$ just holds in the Balanced Equilibrium, $A^D$ is at which dealers buy from and sell to investors at the same probability. Then, there would be just as many $L_0$s and $L_2$s entering the inter-dealer market, in which case small dealers do not trade in the market. For any smaller (larger) $A^D$, as in the lower (upper) subset of the $e$-$A$ space for the Balanced Equilibrium to hold in Figure 1, dealers buy at a smaller (larger) probability while selling at a higher (smaller) probability in the investor-dealer market to result in fewer (more) $L_2$s entering the inter-dealer market to sell than $L_0$s entering the market to buy. Small dealers sell (buy) in equilibrium to eliminate the excess demand (supply) among large dealers.

The proof of the Proposition in the Appendix shows that the condition in Parts (a)-(c) are also the conditions for how the inter-dealer market can clear for the given type of equilibrium in (18)-(21). Recall that all dealers exit the inter-dealer market and enter the investor-dealer market with their respective optimal inventories. A given round of trading with investors afterwards would leave all dealers that have just traded weakly prefer to trade again in the
inter-dealer market to restore their respective optimal inventories. Those dealers who strictly prefer to trade in one direction must, however, be less numerous than dealers who weakly prefer to trade in the other direction, meeting the requirement for market clearing, since in each type of equilibrium, there is one type of dealers indifferent between holding two levels of inventory.

2.7 The Makeup of the Dealer Population and Equilibrium Types

Proposition 2 focuses on the role the asset supply plays in determining equilibrium outcomes. To complete the analysis, we next turn our attention to the role the makeup of the dealer population plays.

To begin, if there were no large dealers, any inter-dealer trades would only be between an \( S_1 \) selling to an \( S_0 \). The equilibrium terms of trade must then be such that the two parties are indifferent between trading and not trading; i.e. \( p = \beta (V_1^{SD} - V_0^{SD}) \) as in the Balanced Equilibrium. For any positive \( n^{LD} \) not up to \( \epsilon \), by Proposition 2, there can still only be a Balanced Equilibrium in which small dealers remain indifferent between trading and not trading. But as \( n^{LD} \) rises up to and above \( \epsilon \), the Selling and the Buying Equilibrium may begin to hold in which small dealers strictly prefer to trade.

The conditions in Proposition 2 may be manipulated to trace out the evolution of equilibrium type as \( n^{LD} \) increases from the smallest admissible value to \( n^D \).

**Corollary 1** Holding fixed \( n^D \), the Balanced Equilibrium begins to hold for

\[
n^{LD} > \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right) \left( A - \frac{e}{\delta} - e \right) - n^D + e.
\]

As \( n^{LD} \) increases from the lower bound in the above towards \( n^D \), the Balanced Equilibrium holds throughout only if \( A - e/\delta = B \), where

\[
\overline{B} \equiv n^D + \frac{n^D}{\mu^{-1} \left( \frac{A^D}{n^D} \right)}
\]

In general, the Balanced Equilibrium changes into the Buying Equilibrium for \( A - \frac{e}{\delta} > \overline{B} \) at

\[
n^{LD} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^D} \right) \left( A - \frac{e}{\delta} - n^D \right),
\]

but into the Selling Equilibrium for \( A - \frac{e}{\delta} < \overline{B} \) at

\[
n^{LD} = A - \frac{e}{\delta} - \frac{n^D}{\mu^{-1} \left( \frac{e}{n^D} \right)}.
\]

When large dealers become relatively more numerous then, the Balanced Equilibrium in general must give way to either the Buying or the Selling Equilibrium. In particular, if the asset supply is relatively abundant as with \( A - e/\delta > \overline{B} \), for the inter-dealer market to clear in the Balanced Equilibrium, \( S_0 \)s buy to eliminate the excess supply among large dealers. But
then when $n^{LD}$ rises up to and $n^{SD}$ falls down to some given levels, the remaining $S_0$s would no longer be sufficiently numerous to fulfill such a role. In this case, equilibrium can only obtain when $L_0$ no longer strictly prefer to sell as when the inter-dealer market price falls from $p = \beta (V_1^{SD} - V_0^{SD})$ to $p = \beta (V_1^{LD} - V_0^{LD})$, at which point the Buying Equilibrium takes hold. Conversely, how the Balanced Equilibrium changes into the Selling Equilibrium when the asset supply is relatively meager as with $A - e/\delta < \mathcal{T}$ is due to how not all $L_0$s can remain as buyers when the remaining $S_1$s would no longer suffice to fill the gap between the supply and demand from large dealers as those dealers make up a large enough fraction of the dealer population.

3 Direction of Trade: Small Dealers Provide Immediacy for Large Dealers

To be sure, in reality, dealers in an OTC market differ in their inventory capacities for various reasons. A major point of interest of our analysis is that the asset supply $A$ plays an important role in determining the direction of trades among dealers having different capacities to hold the asset – a question that cannot be answered by other models of inter-dealer trades in which the asset supply should merely scale the trading volume but not affect the direction of trades.

Specifically, a direct corollary of Proposition 2 is that:

**Corollary 2** Large dealers only sell to but not buy from small dealers for $A > S + e/\delta$, whereas for $A < S + e/\delta$, large dealers only buy from but not sell to small dealers.

For a given asset supply then, trades among small and large dealers in our model take place in one direction only. But which direction it takes can vary with the asset supply. A priori it seems intuitive that large dealers, to the extent that they are able to and indeed tend to hold a larger inventory, should sell to small dealers. In our model, however, this is the case only when the asset supply is relatively abundant – just when small dealers should find it easiest to buy the asset from investors themselves. A dealer is said to provide immediacy to another dealer if the first dealer sells the asset to (buys from) the other dealer at times when it takes longest on average for the latter to buy (sell) the asset in the market as arising from, for example, a meagre (abundant) asset supply. Apparently, large dealers in our model do not tend to provide immediacy for small dealers.

The proof of Proposition 2 shows that for $A < S + e/\delta$, $\eta(\theta_{ID}) > \mu(\theta_{DI})$, whereby dealers are more likely to meet investor-buyers than investor-sellers; i.e., the scarcity of the asset should give rise to more selling opportunities than buying opportunities for dealers. In our model, small dealers sell to large dealers to provide inventory for the latter to sell to investors in this case. For $A > S + e/\delta$, with $\eta(\theta_{ID}) < \mu(\theta_{DI})$, dealers need spare inventory capacity more than inventory in a market with a large asset supply as the abundance of the asset gives rise to more buying opportunities than selling opportunities for dealers. In our model, small dealers buy from large dealers to help them free up capacities to buy from investors. In all, it is the small dealers who provide immediacy for large dealers in our model.

It is well known that inter-dealer trades in many OTC markets can be characterized by a so-called core-periphery trading structure in which a set of dealers, referred to as the core
dealers, are observed to trade with all dealers while the rest, referred to as the peripheral dealers, are observed to trade only with the core dealers but not among themselves (Li and Schürhoff (2014) and Hollifield, Neklyudov and Spatt (2016)). True, assuming that inter-dealer trades take place in a competitive inter-dealer market, our model has no predictions as to the exact identity of the dealer whom a given dealer tends to trade with. But we do know that all small dealers either sell or buy in a given type of equilibrium in which case a given small dealer cannot be buying from or selling to another small dealer. On the other hand, in all three types of equilibrium in our model, there are large dealers who are selling and others who are buying in which case a given large dealer can be trading with another large dealer in one period but then with a small dealer in different time periods. In this way, the large and small dealers in our model behave similarly as the core and the peripheral dealers do, respectively, identified in the empirical studies with regard to the set of dealers they are predicted to trade with.\footnote{One interpretation of a competitive market is that trades are literally centralized and that buyers and sellers do not trade with each other but instead with a Walrasian auctioneer. Of course, in this interpretation, our model cannot be taken to imply any kind of trading structure whatsoever. In our view, such an interpretation is needlessly agnostic. To us, the defining characteristic of perfect competition is that all trades take place at a price that equates demand and supply and that the Walrasian auctioneer story is but one story justifying the assumption. For more sophisticated theoretical foundations, see, for example, Gale (2000).} 

In our model then, the large dealers can be interpreted to play the role of core dealers in a core-periphery trading network, trading with all dealers in the market while the small dealers play the role of peripheral dealers, trading only with the core dealers.\footnote{Li and Schürhoff (2014) find that core dealers tend to hold more assets in inventory. In all three types of equilibrium in our model, the optimal inventory level for a large dealer (who is in the core) is at least weakly higher than that of a small dealer (who is in the periphery).} Under this interpretation, our model predicts that the small peripheral dealers provide immediacy for the large core dealers in general by selling to the latter when it takes a relatively long time for the core dealers to buy from investors themselves but buying from the latter when it is difficult for the core dealers themselves to sell to investors.\footnote{In other network-theoretic models of inter-dealer trades with a core-periphery structure, the core dealers are either identified with dealers that sell to and thus provide inventory for peripheral dealers (Farboodi (2014) and Zhong (2014)), dealers that generally provide immediacy for peripheral dealers by virtue of being more connected to other dealers (Wang (2017)), or dealers that simply tend to trade more frequently (Neklyudov (2015) and Hugonnier et al. (2016)).} 

This implication is counterintuitive but there indeed exists empirical evidence supporting it. Hollifield et al. (2015) report in their Table 6 that the percentage bid-ask spreads peripheral dealers earn when they are the first links of the intermediation chains, buying from an investor for selling to other dealers, is smaller than the percentage spreads they earn when they are the last links, buying from another dealer for selling to an investor. There are no statistically significant differences between the two spreads for core dealers, however.

In our model, when the latter part of the Balanced Equilibrium or the Buying Equilibrium holds as arising from an abundant asset supply, the small peripheral dealers are in the last link of the intermediation chains. And then with an abundant asset supply, the small peripheral dealers would be selling to investors in a slow market (small $\theta_{ID}$) – a market in which it takes a long time on average for a dealer to sell the unit in his inventory. The intermediation services provided by small dealers for large dealers by selling to investors on their behalf in the slow market should then command a high return.
On the other hand, when the Selling Equilibrium or the earlier part of the Balanced Equilibrium holds as arising from a meager asset supply, small dealers are in the first link of the intermediation chains. With the meager asset supply, the small peripheral dealers would be buying from investors in a tight market \((\text{large } \theta_{DI})\), a market populated by a large number of dealer-buyers versus a small number of investor-sellers. In the tight market, where the competition among dealers can be intense, there should only be a small return earned by a dealer from intermediating the sale by an investor since the dealer has a relatively weak bargaining position vis-à-vis an investor having plenty of other meeting opportunities.

Averaging over markets with various levels of asset supply, small dealers in our model indeed should earn a higher markup when they buy from other dealers for selling to investors than when they buy from investors for selling to other dealers, just as what Hollifield et al. (2015) find in their empirical analysis. To confirm this conjecture, we calculate and then plot the markup

\[
\rho_{DB} \equiv \frac{p - p_{IS}}{p_{IS}}
\]

small dealers earn by intermediating the sale by investors in the Selling Equilibrium and the earlier part of the Balanced Equilibrium and the markup

\[
\rho_{DS} \equiv \frac{p_{IB} - p}{p}
\]

they earn by intermediating the sale by other dealers in the latter part of the Balanced Equilibrium against \(A\) in Figure 2. When the Buying Equilibrium holds, \(\rho_{DS} = \infty\) with \(p = 0\), and so is left out of the plot. By changing the normalization that low-valuation investors and dealers place a positive value, instead of zero, on holding a unit of the asset, \(p\) would stay positive in any equilibrium while the basic forces in our model should still contrive to give rise to a relatively large \(\rho_{DS}\) in the Buying Equilibrium.

The tendencies that \(\rho_{DB}\) should be low for small \(A\) and \(\rho_{DS}\) should be high for large \(A\) arise out of the influences of market tightness on agents’ bargaining positions and should be common to most models of frictional OTC markets. Models that are built to be consistent with core dealers providing immediacy for small dealers would then likely to predict just the opposite of the findings in Hollifield et al. (2015).

Last but not least, we should mention that our model can also be consistent with the finding in Hollifield et al. (2015) that the central dealers do not tend to earn higher or lower percentage markups between buying from and selling to customers in a dealer chain. The large core dealers in our model are dealer-buyers as well as dealer-sellers in all three types of equilibrium. For each \(A\) then, there are large dealers earning \(\rho_{DS}\) as well as \(\rho_{DB}\). In Figure 3,

\footnote{We formally show that both \(\theta_{ID}\) and \(\theta_{DI}\) are (weakly) decreasing and continuous in \(A\) in Proposition 4a below.}

\footnote{The numerical analyses assume \(\eta(\theta) = 1 - e^{-\theta}, \epsilon = 0.8, \delta = 0.1, \beta = 0.95, n^{LD} = 0.5, n^{SD} = 1\) and \(A\) varying from 9.86 – the lower bound for the Selling Equilibrium to hold – and up. The equations for \(p, \theta_{DI}, \) and \(\theta_{ID}\) for the Selling Equilibrium are given by (74), (83), and (85), respectively. The equations for \(p, \theta_{ID}, \) and \(\theta_{DI}\) in the Balanced Equilibrium are given by (76), (96), and (97), respectively. The equations for \(p_{IS}\) and \(p_{IS}\) for the Selling and Balanced Equilibria are stated in Lemma A1 in the Appendix. The discontinuities in \(\rho_{DS}\) and \(\rho_{DB}\) are due to \(p\) changing its anchor from one to another indifference conditions as one equilibrium type changes to another. See Proposition 4c below.}
Figure 2: Small dealers’ percentage bid-ask spreads

Figure 3: Large dealers’ percentage bid-ask spreads
we plot $\rho_{DS}$ and $\rho_{DB}$ for all levels of $A$. Although large dealers earn a higher $\rho_{DS}$ than $\rho_{DB}$ in the Balanced and in the Buying Equilibria, they earn a higher $\rho_{DB}$ instead in the Selling Equilibrium. Averaging over the $A$ for assets included in a given sample, there can be no statistically significant differences between the observed $\rho_{DS}$ and $\rho_{DB}$ for the large dealers.

4 Robustness of the Equilibrium Features

The model we studied in this paper is admittedly a very special model, with numerous important simplifying assumptions. In this section, we explain how our major results regarding trading directions should survive three generalizations that are most warranted. Notwithstanding the discussions below falling short of full-fledged formal analyses in various occasions for the interest of brevity, the claims not derived from formal proofs in the following all appear to be intuitive extensions of the corresponding results in the main model. All lemmas and Proposition 3 below are formal results though the proofs of which are in the Appendix.

4.1 Larger Inventory Capacity for Small Dealers

Perhaps it seems trivial that, in our model, small dealers, in having just one unit of inventory capacity, never gain from trading among themselves. The question then is if and how the trading directions in Corollary 2 survive the generalization where small dealers each possess more than a unit of inventory capacity and thereby may gain by trading with one another.

Consider, in particular, that the small dealers each possess a two-unit inventory capacity, while the large dealers each possess a three-unit inventory capacity. These larger capacities are relevant only if a dealer may buy and sell up to two units of the asset in a period. The simplest extension is to assume that there are two types of investors – small and large, where the former, comprising a fraction $\alpha$ of the investor population, may each hold either zero or one unit, whereas the latter, comprising the rest of the investor population, may each hold either zero or two units, and that dealers meet investors randomly independent of dealers’ types. In this environment, a dealer-seller (-buyer) holding a one-unit inventory (spare capacity) can only sell to (buy from) small investors who buy (sell) one unit of the asset at a time, whereas a dealer-seller (-buyer) holding at least a two-unit inventory (spare capacity) can sell to (buy from) large investors, who buy (sell) two units at a time, as well as small investors.

Now, a ranking of the marginal benefits of inventory similar to that in Proposition 1 should remain – an additional unit of the asset should be valued higher by a large dealer than by a small dealer at the same initial level of inventory for the two dealers since the former would retain a greater spare capacity for future buying needs than the latter in using up a unit of capacity for acquiring the unit. On the other hand, the large dealer should value an additional unit of the asset less than the small dealer when they start with the same spare capacity, as the former has a larger initial inventory than the latter beforehand. The ranking of the marginal benefits of inventory in Proposition 1 may then be generalized to

$$V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD} \geq V_{2}^{LD} - V_{1}^{LD} \geq V_{2}^{SD} - V_{1}^{SD} \geq V_{3}^{LD} - V_{2}^{LD},$$

where the inter-dealer market clears in general only at $p$ equal to $\beta$ times one of the above marginal values. We next proceed to inquire how the trading directions between small and
large dealers remain persistent while small dealers trade to provide immediacy for large dealers.

**Case 1a** $p = \beta (V_{1}^{LD} - V_{0}^{LD})$ The buyers in the inter-dealer market are comprised of a fraction of $L_0$s and the sellers are $S_1$s, $S_2$s, $L_2$s, and $L_3$s.

**Case 1b** $p = \beta (V_{1}^{SD} - V_{0}^{SD})$ with a fraction of $S_1$s selling in the inter-dealer market
The buyers in the inter-dealer market are all of $L_0$s and the sellers are a fraction of $S_1$s, and all of $S_2$s, $L_2$s, and $L_3$s.

For $p$ to settle at the highest or the second highest marginal values of inventory, the two cases above should hold for the smallest $A$. Given that all small dealers who trade in the inter-dealer market ($S_1$s and $S_2$s) sell and they sell to $L_0$s, small dealers trade to provide immediacy for large dealers and the trading direction between small and large dealers is persistent.

**Case 2** $p = \beta (V_{2}^{LD} - V_{1}^{LD})$ with a fraction of $S_0$s buying in the inter-dealer market
The buyers in the inter-dealer market are all of $L_0$s, and a fraction of $S_0$s, and the sellers are $L_2$s, $S_2$s, and $L_3$s. When this type of equilibrium first starts to hold, the fraction of $S_0$s who buy is arbitrarily close to zero. When this equilibrium turns into the equilibrium at $p = \beta (V_{2}^{LD} - V_{1}^{LD})$ so that all $S_0$s are buying, as we shall demonstrate below in the next case, there would be as many $S_0$s as $S_2$s. In between, we conjecture that there remains fewer $S_0$ buyers than $S_2$ sellers. Then, on balance, small dealers are selling to and thus are still providing immediacy for large dealers at a $p$ that should hold for relatively small $A$. The trading direction, though not perfectly, is largely persistent.

**Case 3** $p = \beta (V_{2}^{LD} - V_{1}^{LD})$ The buyers in the inter-dealer market are all of $L_0$s, $S_0$s, and possibly a fraction of $L_1$s. The sellers are $S_2$s, $L_3$s, and possibly a fraction of $L_2$s. Given that all small dealers leave the inter-dealer market with one unit of inventory whereas large dealers do so with either one or two units of inventory, when the investor-dealer market opens, all dealers are dealer-sellers as well as dealer-buyers. All this can be shown to imply that:

**Lemma 3** (a) $\eta(\theta_{ID}) = \mu(\theta_{DI})$, (b) $m_{0}^{SD} = m_{2}^{SD}$, and (c) $m_{0}^{LD} = m_{3}^{LD}$.

Part (a) of the Lemma says that it is equally likely for dealers to meet investor-buyers and investor-sellers in which case a unit of inventory would not be any more or less useful than a unit of inventory capacity to dealers. There should then be no particular role to be played by small dealers in providing immediacy for large dealers. Parts (b) and (c) together say that what follows next is that there are equal measures of inframarginal buyers ($m_{0}^{SD} + m_{0}^{LD}$) and sellers ($m_{2}^{SD} + m_{3}^{LD}$), whereby the indifferent traders $L_1$s and $L_2$s do not trade in the inter-dealer market. This equilibrium then corresponds to the mid-point of the Balanced Equilibrium in the basic model at which $\eta(\theta_{ID}) = \mu(\theta_{DI})$ and that the indifferent traders $S_0$s and $S_1$s do not trade. Here, with small dealers selling and buying in the inter-dealer market equally numerous, they indeed do not tend to provide immediacy for large dealers. For $\eta(\theta_{ID}) = \mu(\theta_{DI})$ exactly, $A$ must be at just one particular level, as in the basic model.
Case 4  \( p = \beta (V_{SD}^2 - V_{SD}^1) \) with a fraction of \( S_2 \)s selling in the inter-dealer market

This is the mirror opposite of case 2. Small dealers buy from more than they sell to large dealers, helping large dealers free up inventory capacities and providing immediacy for them on balance with \( p \) settling at a relatively low level as arising from relatively large \( A \).

Case 5  \( p = \beta (V_{SD}^2 - V_{SD}^1) \) with a fraction of \( S_1 \)s buying in the inter-dealer market or \( p = \beta (V_{LD}^3 - V_{LD}^2) \) The is the mirror opposite of case 1. Small dealers buy from large dealers only, providing immediacy for large dealers, with \( p \) settling at the lowest levels as arising from the largest \( A \).

The above suggests that the result that small dealers provide immediacy should also generalize to where there are more than two inventory capacities, as similar mechanisms should be operative to give rise to smaller-capacity dealers selling (buying) the asset to (from) larger-capacity dealers when the asset supply is small (large).

4.2 Frictional inter-dealer market

In reality, the inter-dealer market is better described as a decentralized market as suggested by the findings in Li and Schürhoff (2014) and Henderschott, Li, Livdan and Schürhoff (2016), where it takes time and effort for a dealer to find a counterparty to trade with, in which case dealers, by all means, have incentives to manage inventory for future trading needs. By assuming dealers only have periodic, instead of continuous, access to the competitive inter-dealer market, the dealers in our model likewise have incentives to manage inventory. Where the incentives are similar, many features of the equilibrium in the present model should survive in an arguably richer model of a frictional inter-dealer market.

In the following, we report the results of our analysis of a model of a frictional inter-dealer market, which is otherwise identical to the main model of the paper, except that the model is set in continuous time as it is a more convenient setting to analyze a model in which both the investor-dealer and the inter-dealer markets are decentralized. In the revised model, we continue to assume that the search and matching in the investor-dealer market takes place in two market segments, with respective market tightness \( \theta_{ID} \) and \( \theta_{DI} \). The inter-dealer market is frictional, however, in which a given dealer meets another randomly selected dealer at a fixed rate \( \alpha \) per unit of time, and where the terms of trade between two dealers are determined by Nash Bargaining, as are prices in the investor-dealer market. The value functions and equilibrium conditions are presented in Appendix 7.1. The pricing equations are standard and are omitted for brevity.

We first verify that:

Lemma 4  Any investor-dealer match yields a non-negative surplus in any equilibrium in which both small and large dealers are active.

This Lemma generalizes Lemma 1 for a competitive inter-dealer market. The proof of the Lemma proceeds with the idea that first, if a given dealer-seller \( DS \) chooses not to sell to investor-buyers (\( IB \)), he must then sell to other dealers, say dealer \( d \), for otherwise the unit will never be passed on to an \( IB \), in which case it would never be optimal for dealer \( DS \) to
acquire the unit in the first place. Now, if it is optimal for dealer \( d \) to sell to an \( IB \), it must be optimal for dealer \( DS \) to sell to the \( IB \) as well – there cannot be any greater surplus of trade for the unit to pass to another dealer before the unit is sold to an \( IB \). A similar argument explains how a given dealer possessing spare capacity must find it optimal to buy from an investor-seller should the opportunity arises if the dealer should stay active in equilibrium at all. Given the Lemma, we then proceed to show that the counterpart to Proposition 1 for the competitive inter-dealer market holds.\(^{18}\)

**Proposition 3** \( V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD} \geq V_{2}^{LD} - V_{1}^{LD} \) in any active equilibrium. The two equalities are strict unless the surplus for the \( IB-L_{1} \) match,

\[
z_{IB,L_{1}} \equiv U_{H}^{ON} - U^{B} - (V_{1}^{LD} - V_{0}^{LD})
\]

is equal to zero.

By Proposition 3 and if the inequalities are strict, among the respective surpluses for the possible inter-dealer trades, only

\[
z_{L_{0},S_{1}} \equiv V_{1}^{LD} - V_{0}^{LD} + V_{0}^{SD} - V_{1}^{SD} > 0,
\]

\[
z_{S_{0},L_{2}} \equiv V_{1}^{SD} - V_{0}^{SD} + V_{1}^{LD} - V_{2}^{LD} > 0,
\]

\[
z_{L_{0},L_{2}} \equiv V_{1}^{LD} - V_{0}^{LD} + V_{1}^{LD} - V_{2}^{LD} > 0,
\]

whereas all other inter-dealer trades yield negative surpluses. Hence, exchanges between an \( L_{0} \) and an \( S_{1} \), between an \( S_{0} \) and an \( L_{2} \), and between an \( L_{0} \) and an \( L_{2} \) exhaust all profitable exchanges among dealers, as in the competitive inter-dealer market model, from which a core-periphery trading structure emerges. In all, the core-periphery trading structure in our model does not hinge on a Walrasian inter-dealer market but is a generic feature of models assuming dealers having imperfect access to inter-dealer trades, however modeled, and that their inventory capacities differ.

With a competitive inter-dealer market, we have shown that, in Corollary 2, small dealers only sell to (buy from), but not buy from (sell to), large dealers when the asset supply is relatively meagre (abundant). Apparently, such implications no longer hold with a frictional inter-dealer market as both types of trade, each yielding a positive surplus, would take place in equilibrium for all levels of asset supply. Even so, a weaker version of the Corollary can hold if the \( L_{0}-S_{1} \) trades (small dealers’ sales to large dealers) tend to dominate the \( S_{0}-L_{2} \) trades (small dealers’ purchases from large dealers) when the asset supply is relatively meagre and vice versa.\(^{19}\)

---

\(^{18}\)All notations for the revised model have the same meanings as for the main model.

\(^{19}\)The equilibrium conditions turn out to be highly nonlinear in equilibrium objects, whereby it does not seem possible to analytically solve the model and to derive conditions on model fundamentals for the existence and uniqueness of equilibrium. Assuming a Walrasian inter-dealer market simplifies considerably and enables us to derive a rich set of analytical results.
In our numerical analyses,\textsuperscript{20} we find that the volume of $L_0$-$S_1$ trades, given by\textsuperscript{21}

\begin{equation}
SD_s = n_1^{SD} \frac{n_0^{LD}}{n_D}
\end{equation}

is decreasing in $A$, whereas the volume of $S_0$-$L_2$ trades, given by

\begin{equation}
SD_b = n_0^{SD} \frac{n_2^{LD}}{n_D}
\end{equation}

is increasing in $A$ as shown in Figure 4, and that $SD_s$ dwarfs (is dwarfed by) $SD_b$ for small (large) $A$.\textsuperscript{22} Hence, even though small dealers do buy from and sell to large dealers for all levels of $A$, they overwhelmingly sell to provide inventory for large dealers for small $A$ but buy from them to provide capacity for large dealers for large $A$ just as in the competitive inter-dealer market model.

The first order effect of an increase in $A$ is that there should be more dealers searching with a full inventory (larger $n_1^{SD}$ and $n_2^{LD}$) and fewer dealers searching with an empty inventory (smaller $n_0^{SD}$ and $n_0^{LD}$) for the additional units of the asset to be in circulation. A priori then, by (36) and (37), respectively, it seems that either $SD_s$ or $SD_b$ may go up or down with $A$.

There are also second order effects arising from inter-dealer trading, however. Specifically, with more dealers searching with a full inventory who may sell to dealers searching with an

\textsuperscript{20}In the numerical example, we assume $\eta(\theta) = \theta^{0.5}$, $\alpha = 1$, $e = 0.1$, $d = 0.05$, and $\{n_{SD}, n_{LD}\} = \{0.6, 0.1\}$. A unique steady-state equilibrium exists for $A \in [2.77, 3.61]$.

\textsuperscript{21}An $S_1$ meets another dealer at the rate $\alpha$ and a fraction $n_0^{LD}/n_D$ of those dealers are $L_0$. The equation for $SD_b$ is constructed similarly.

\textsuperscript{22}These results are from assuming a large $n_{SD}^{SD} (= 0.6)$ relative to $n_{LD}^{LD} (= 0.1)$. As checks for robustness, we find the same qualitative results hold with $\{n_{SD}, n_{LD}\} = \{0.1, 0.6\}$ and $\{0.35, 0.35\}$.}

---

Figure 4: Small-large dealers trade volume
empty inventory, inter-dealer trading should reinforce the decline in the measures of the latter
group of dealers even further. Since \( L_0 \)s buy when they meet \( S_1 \)s as well as \( L_2 \)s, whereas \( S_0 \)s
buy only when they meet \( L_2 \)s, this second order effect would be felt more on \( n^{LD}_0 \) than on
\( n^{SD}_0 \). Other things equal, there are stronger forces for \( SD_s \) to go down with \( A \) than for \( SD_b \).

On the other hand, with fewer dealers searching with an empty inventory who may buy
from dealers searching with a full inventory, inter-dealer trading should similarly reinforce the
increase in the measures of the latter group of dealers even further. Since \( L_2 \)s sell when they
meet \( L_0 \)s as well as \( S_0 \)s, while \( S_1 \)s sell only when they meet \( L_0 \)s, this second order effect should
be felt more on \( n^{LD}_2 \) than on \( n^{SD}_1 \). Other things equal, there are stronger forces for \( SD_b \) to go
up with \( A \) than for \( SD_s \). All together then, where there are stronger forces for \( SD_s \) to go down
and stronger forces for \( SD_b \) to go up with \( A \), on balance, \( SD_s \) should only become decreasing
and \( SD_b \) become increasing in \( A \).

4.3 Matching Opportunity

If each large dealer can hold up to two units in inventory and may possess up to two units of
spare inventory capacity, perhaps a more natural assumption is that they can meet up to two
investor-buyers and two investor-sellers in each period. We shall demonstrate below how the
ranking of the marginal benefits of inventory in Proposition 1 can be left intact.

First, if a large dealer has up to two matching opportunities with investor-buyers and with
investor-sellers, respectively, a reasonable matching technology should be such that

1. the probability that a large dealer meets at least one investor-buyer is weakly higher than
   the probability that a small dealer meets one investor-buyer,

2. the probability that a large dealer meets at least one investor-seller is weakly higher than
   the probability that a small dealer meets one investor-seller.

If, in addition, the matching technology exhibits diminishing returns in the sense that

3. the probability that a large dealer meets two investor-buyers is weakly lower than the
   probability that a small dealer meets one investor-buyer,

4. the probability that a large dealer meets two investor-sellers is weakly lower than the
   probability that a small dealer meets one investor-seller,

then the ranking in Proposition 1 remains.\(^{23}\)

The arguments are as follows. First, consider the costs and benefits of acquiring the first
unit of inventory in the inter-dealer market for the two types of dealers. Filling up the first unit
of capacity in the inter-dealer market is costly to a small dealer as long as he shall meet one

\(^{23}\)These assumptions are easily satisfied if the two matching outcomes for the large dealer are independent
events. In this case, the probability that a large dealer meets at least one investor-buyer is

\[
1 - (1 - \eta(\theta_{ID}))^2 = \eta(\theta_{ID})(2 - \eta(\theta_{ID})) > \eta(\theta_{ID}),
\]

where the far-right term is the probability that the small dealer meets one investor-buyer. On the other hand,
the probability that a large dealer meets as many as two investor-buyers is \( \eta(\theta_{ID})^2 < \eta(\theta_{ID}) \).
investor-seller in the next period but is costly to a large dealer only if he meets two investor-sellers in the next period since if a large dealer meets only one investor-seller, he still has capacity to buy. By (4), the expected cost is higher for the small dealer. The expected benefit is higher for the large dealer - if (1) holds, the large dealer can sell the unit with weakly higher probability. Then, \( V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD} \) should follow. The inequality should be strict if either one of the relation in (1) or (4) is strict.

Next, consider the costs and benefits of utilizing the last unit of spare capacity for the two types of dealers. Exhausting one’s capacity is costly to a dealer, large or small, as long as the dealer shall meet one or more investor-seller in the next period. By (2), the expected cost is higher for the large dealer. A small dealer benefits from the additional unit of inventory if he meets one investor-buyer while a large dealer benefits only if he meets as many as two investor-buyers. If (3) holds, the expected benefit is higher for the small dealer. Then, \( V_{1}^{SD} - V_{0}^{SD} \geq V_{2}^{LD} - V_{1}^{LD} \) should follow. The inequality should be strict if either one of the relation in (2) or (3) is strict.

5 Comparative Statics

Having shown how the major results of the model hold in more general settings, we now return to the basic model and study the model’s comparative statics.

5.1 Asset Supply

Market Tightness and Turnover  In Section 3, we remarked that dealers should find it easier to buy but more difficult to sell in a market with more abundant asset supply. We state the formal results in the following proposition.

Proposition 4a (i) For \( e < n^{LD} \), as \( A \) increases from \( B_{L} + e/\delta \) at which the Selling Equilibrium first holds, \( \partial \theta_{DI}/\partial A = 0 \) and \( \partial \theta_{ID}/\partial A < 0 \). Once \( A \) reaches \( B_{M} + e/\delta \) at which the Balanced Equilibrium begins to hold, \( \partial \theta_{DI}/\partial A < 0 \) and \( \partial \theta_{ID}/\partial A < 0 \). Finally, when \( A \) rises up to and above \( B_{L} + e/\delta \) at which the Buying Equilibrium holds, \( \partial \theta_{DI}/\partial A < 0 \) and \( \partial \theta_{ID}/\partial A = 0 \). In the transition from one equilibrium type to another, \( \theta_{DI} \) and \( \theta_{ID} \) are continuous. (ii) For \( e \in \left[ n^{LD}, n^{LD} + n^{SD}/2 \right) \) and that \( A > B_{M} + e/\delta \) at which the Balanced Equilibrium holds, \( \partial \theta_{DI}/\partial A < 0 \) and \( \partial \theta_{ID}/\partial A < 0 \).

Proposition 4a implies that indeed, as \( A \) rises, a dealer-buyer meets an investor-seller at a (weakly) higher probability \( \mu(\theta_{DI}) \) while a dealer-seller meets an investor-buyer at a (weakly) lower probability \( \eta(\theta_{ID}) \).

Inter-dealer Trading Volume  In the inter-dealer market, trades are driven by the inframarginal buyers’ or sellers’ desire to rebalance inventories. The trading volume \( (TV) \) in the Selling, Balanced, and the Buying Equilibria are then given by, respectively,

\[
TV = m_{1}^{SD} + m_{2}^{LD},
\]
\[
TV = \begin{cases} 
  m_0^{LD} & A \leq S + \frac{e}{S} \\
  m_2^{LD} & A \geq S + \frac{e}{S},
\end{cases}
\]

\[
TV = m_0^{SD} + m_0^{LD}.
\]

**Proposition 4b** The inter-dealer market trading volume changes non-monotonically with \( A \), as depicted in the table below.

<table>
<thead>
<tr>
<th>Selling Equilibrium</th>
<th>Balanced Equilibrium</th>
<th>Buying Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( A \leq S + \frac{e}{S} )</td>
<td>small dealers sell</td>
</tr>
<tr>
<td></td>
<td>( A \geq S + \frac{e}{S} )</td>
<td>small dealers buy</td>
</tr>
<tr>
<td>( \frac{\partial TV}{\partial A} &gt; 0 )</td>
<td>( \frac{\partial TV}{\partial A} &lt; 0 )</td>
<td>( \frac{\partial TV}{\partial A} &gt; 0 )</td>
</tr>
<tr>
<td></td>
<td>( \frac{\partial TV}{\partial A} &lt; 0 )</td>
<td>( \frac{\partial TV}{\partial A} &lt; 0 )</td>
</tr>
</tbody>
</table>

For \( e < n^{LD} \), the trading volume changes continuously as one equilibrium type changes to another, peaking at \( TV = e \left( 1 - \frac{e}{n^{SD}} \right) \), when the Selling Equilibrium turns into the Balanced Equilibrium and when the Balanced Equilibrium turns into the Buying Equilibrium.

The inter-dealer market is most active then when the asset supply is at a relatively low level but not at the lowest level or at a relatively high level but not at the highest level. The prediction is in stark contrast to what we would obtain in maybe any other models of inter-dealer trades, where a higher asset supply should merely scale up the trading volume. The non-monotonicity in our model is due to the interactions of changing market tightness and optimal dealers’ inventories as brought about by variations in \( A \). In particular, say, to begin with, the asset supply is at the lowest level for which very few dealers can buy from investors in a very tight market, whereby, trades in the inter-dealer market, driven by the successful dealer-buyers’ need to dispose the inventories they acquire from investors, can only be few and far between. As the asset supply rises, successful dealer-buyers become more numerous, whereby inter-dealer trades begin to pick up. When the asset supply, still at a relatively low level, reaches the point at which the Balanced Equilibrium takes hold, trading in the inter-dealer market changes into being driven by the successful dealer-sellers replenishing their inventories in the market. As \( A \) increases further, there are fewer successful dealer-sellers and such needs to replenish inventories weaken. How inter-dealer trades are least numerous when \( A \) is at the highest level, begin to pick up as \( A \) first goes down, but then declines once \( A \) falls down to a low enough level can be understood similarly.

**Inter-dealer Trading Prices** Proposition 4a shows that dealers find it easier to buy from and harder to sell to investors as \( A \) increases. If dealers who have bought from investors tend to sell and dealers who have sold to investors tend to buy afterwards in the inter-dealer market, an increase in \( A \) should be followed by an increase in supply and a decline in demand in the inter-dealer market and a concomitant decline in the inter-dealer market price. Besides, when one equilibrium type turns into another, the inter-dealer market price \( p \) changes its anchor
from one indifference condition to another. As such, a minute change in the asset supply can cause a catastrophic change in $p$.

**Proposition 4c** (i) For $e < n^{LD}$, as $A$ increases from $B_L + e/\delta$ at which the Selling Equilibrium first holds, $p$ is continuously decreasing in $A$. Once $A$ reaches $B_M + e/\delta$ at which the Balanced Equilibrium begins to hold, there will be a discrete fall in $p$, followed by further continuous decreases as $A$ increases further. Finally, when $A$ rises up to and above $B_L + e/\delta$ at which the Buying Equilibrium holds, there will be another discrete fall in $p$ all the way to zero. (ii) For $e \in \left[ n^{LD}, n^{LD} + \frac{n^{SD}}{2} \right]$ and that $A > B_M + e/\delta$ at which the Balanced Equilibrium begins to hold, $p$ is continuously decreasing in $A$ throughout.

### 5.2 Measure of Large Dealers $n^{LD}$

In the comparative statics exercises below, we hold constant the total measure of dealers and vary the measure of large dealers. To the extent that there is least diversity among dealers when all dealers are small dealers and when all dealers are large dealers, the comparative statics with respect to $n^{LD}$ we present below can also be interpreted as the impacts of the diversity of dealers in the dealer population on equilibrium outcomes.

**Market Tightness and Turnover** If more of the dealers are large dealers possessing a two-unit inventory capacity, there will be a greater overall inventory capacity among dealers. Then, first of all, there will tend to be more dealer-buyers. Furthermore, when more dealers are buying from investors, dealers’ overall inventory holding tends to increase as well, giving rise to there being more dealer-sellers.

**Proposition 5a** Holding fixed $n^D$, as $n^{LD}$ increases from the smallest admissible value for which the Balanced Equilibrium holds, $\partial \theta_{DI} / \partial n^{LD} > 0$ and $\partial \theta_{ID} / \partial n^{LD} < 0$. If and when the Balanced Equilibrium gives way to the Buying Equilibrium, $\partial \theta_{DI} / \partial n^{LD} > 0$ and $\partial \theta_{ID} / \partial n^{LD} = 0$. On the other hand, if and when the Balanced Equilibrium gives way to the Selling Equilibrium, $\partial \theta_{DI} / \partial n^{LD} = 0$ and $\partial \theta_{ID} / \partial n^{LD} = 0$. The two market tightness are continuous at the point at which the Balanced Equilibrium turns into either the Buying or the Selling Equilibrium.

The substantive implication of Proposition 5a is that the investor-sellers’ matching rate $\eta(\theta_{DI})$ and the investor-buyers’ matching rate $\mu(\theta_{ID})$ are both weakly increasing in $n^{LD}$. In this way, a market with relatively more large dealers functions better at transferring units of the asset from low- to high-valuation investors.

**Inter-dealer Trading Volume** While the Balanced Equilibrium holds, the trading volume in the inter-dealer market is given by $\max\{m_0^{LD}, m_2^{LD}\}$. Other things equal, there would simply be more dealers searching as $L_0$s and $L_2$s as large dealers make up a greater fraction of the dealer population. Meanwhile, with more dealer-buyers and dealer-sellers searching for investors to trade with, each dealer may only buy or sell at a lower probability to result in more large dealers remaining as $L_0$s and $L_2$s at the closing of each round of investor-dealer trades.
While the Buying Equilibrium holds, the trading volume equals \( m_0^{SD} + m_0^{LD} \). As small dealers are replaced one-for-one by large dealers, dealers leaving the investor-dealer market with an empty inventory should fall in numbers since the large (but not the small) dealers may replenish any inventories they sell to investors in the same period of time by buying from other investors. When the Selling Equilibrium holds, a similar mechanism is at work to result in \( TV \) becoming decreasing in \( n^{LD} \).

**Proposition 5b**  
Holding \( n^D \) fixed, the trading volume in the inter-dealer market is increasing in \( n^{LD} \) while the Balanced Equilibrium holds. Once the Buying or the Selling Equilibrium takes hold, the trading volume becomes decreasing in \( n^{LD} \). \( TV \) is continuous at where the Balanced Equilibrium turns into either the Buying or the Selling Equilibrium and reaches the highest level equal to \( e (1 - \frac{e}{n^P}) \) at the point of transition.

Proposition 5b shows that for any level of asset supply, the inter-dealer market is least active when there is little diversity in the dealer population with \( n^{LD} \) either at the lowest or at the highest level. With more diversity as when \( n^{LD} \) is at some intermediate level, the market becomes more active. Trading in the inter-dealer market in our model then is driven more by the heterogeneity of dealers than by some dealers possessing more than a unit of inventory capacity.

**Inter-dealer Trading Prices**

**Proposition 5c**  
Holding fixed \( n^D \), as \( n^{LD} \) rises, if and when the Balanced Equilibrium gives way to the Buying Equilibrium, \( p \) falls by a discrete amount down to zero; if and when the Balanced Equilibrium gives way to the Selling Equilibrium, \( p \) jumps up by a discrete amount and stays at a given level thereafter for all \( n^{LD} \).

During which the Balanced Equilibrium holds, we find that, through a set of numerical analysis,\(^{24}\) \( p \) is decreasing in \( n^{LD} \), both when the Balanced Equilibrium will turn into the Selling Equilibrium and when it will turn into the Buying Equilibrium. A corollary is that \( p \) can be non-monotonic with respect to increase in \( n^{LD} \) when the Balanced Equilibrium turns into the Selling Equilibrium.

**Investor-Dealer Market Prices**  
For brevity, in Propositions 4c and 5c, we have not extended the analysis to also checking how prices in the investor-dealer market may vary with \( A \) and \( n^{LD} \). In Propositions A1 and A2 and the ensuing discussions in Appendix 7.2, we show that the dealers’ ask and bid prices in the market do turn out to vary with \( A \) and \( n^{LD} \) in the just the same ways that the inter-dealer market price does.

6 Conclusion

In this paper, by means of a parsimonious extension of the standard asset market search model, we study inter-dealer trades among heterogeneous dealers in OTC markets motivated

\(^{24}\)For brevity, the details of the numerical analysis, which are available upon request from the authors, are omitted.
by inventory risk concerns. We depart from earlier inventory risk models by all assuming that
all traders are risk neutral. Even so, the dealers benefit from trading among one another
to eliminate the risks of carrying an insufficient inventory and an insufficient spare inventory
capacity for their trading needs with investors.

The model yields a rich set of testable implications. First and foremost, our analysis shows
that the apparently obvious notion that dealers who are able to and indeed tend to hold a larger
inventory should provide inventory for dealers having a smaller inventory capacity only holds
up when inventory is relatively abundant. But at such times, those smaller-capacity dealers
should need the inventory the least. In contrast, in our model, it is the latter group of dealers
who trade to provide immediacy for the larger-capacity dealers, selling to them when they
need inventory the most and buying from them when they need the spare capacity the most.
If the large dealers are interpreted as the core dealers and the small dealers are interpreted
as the peripheral dealers in a core-periphery trading network, our analysis suggests that the
peripheral dealers trade to provide immediacy for the core dealers, contrary to the common
perception of the roles the two types of dealers should play in inter-dealer trading.

We show in Appendix 7.3 that such features of equilibrium inter-dealer trades actually help
attain constrained efficiency. In the planning optimum, inventories are allocated to dealers to
enable high-valuation investors to acquire the asset most rapidly and to enable units of the
asset to be transferred from low-valuation investors to dealers the quickest, thereby facilitating
the eventual sales to the high-valuation investors. In the competitive inter-dealer market,
inventories and spare capacities are allocated to dealers who value them the most – the very
dealers who have the best use of them for trading with investors. Perhaps not surprisingly, the
equilibrium allocations coincide with the constrained optimum allocations. More interestingly,
our analysis suggests that for efficiency, the small peripheral dealers should indeed trade to
provide immediacy for the large core dealers.
7 Appendix

7.1 The Frictional Inter-Dealer Market Model

The model is set in continuous time in which a dealer meets another randomly selected dealers at the rate $\alpha$ per time unit. All other notations have the same meanings as for the main model.

7.1.1 Value Functions

To define the value functions, we rule out a priori any exchanges between two dealers that merely result in the two dealers concerned switching states as such exchanges cannot give rise to a positive surplus.

Small dealers An $S_0$, who can only buy, meets an investor-seller at the rate $\mu(\theta_{DI})$ and another dealer at the rate $\alpha$. Among all dealers that the $S_0$ may meet, there can be a potentially profitable exchange only if the counterparty is an $L_1$ or an $L_2$. By Lemma 4, all investor-dealer trades yield non-negative surpluses. Then,

$$rV_{0}^{SD} = \mu(\theta_{DI}) (V_{1}^{SD} - V_{0}^{SD} - p_{S_0,I_0}) + \alpha \left\{ \frac{n_{1}^{LD}}{n_{D}} \max \left\{ -p_{S_0,L_1} + V_{1}^{SD} - V_{0}^{SD}, 0 \right\} + \frac{n_{2}^{LD}}{n_{D}} \max \left\{ -p_{S_0,L_2} + V_{1}^{SD} - V_{0}^{SD}, 0 \right\} \right\}. $$

An $S_1$, who can only sell, meets an investor-buyer at the rate $\eta(\theta_{ID})$. The $S_1$ may also sell to an $L_0$ or an $L_1$. Then,

$$rV_{1}^{SD} = \eta(\theta_{ID}) (p_{I_{B},S_1} + V_{0}^{SD} - V_{1}^{SD}) + \alpha \left\{ \frac{n_{0}^{LD}}{n_{D}} \max \left\{ p_{I_{0},S_1} + V_{0}^{SD} - V_{1}^{SD}, 0 \right\} + \frac{n_{1}^{LD}}{n_{D}} \max \left\{ p_{I_{1},S_1} + V_{1}^{SD} - V_{0}^{SD}, 0 \right\} \right\}. $$

Large dealers An $L_0$ may buy from an investor-seller, an $S_1$, or an $L_2$. Then,

$$rV_{0}^{LD} = \mu(\theta_{DI}) (V_{1}^{LD} - V_{0}^{LD} - p_{L_0,I_0}) + \alpha \left\{ \frac{n_{1}^{SD}}{n_{D}} \max \left\{ -p_{L_0,S_1} + V_{1}^{LD} - V_{0}^{LD}, 0 \right\} + \frac{n_{2}^{LD}}{n_{D}} \max \left\{ -p_{L_0,L_2} + V_{1}^{LD} - V_{0}^{LD}, 0 \right\} \right\}. $$

An $L_1$ may buy from an investor-seller and sell to an investor-buyer. Among dealers, he may sell to an $S_0$, buy from an $S_1$, and either buy from or sell to another $L_1$. Then,

$$rV_{1}^{LD} = \mu(\theta_{DI}) (V_{2}^{LD} - V_{1}^{LD} - p_{L_1,I_0}) + \eta(\theta_{ID}) (p_{I_{B},L_1} + V_{0}^{LD} - V_{1}^{LD}) + \alpha \left\{ \frac{n_{0}^{SD}}{n_{D}} \max \left\{ p_{S_0,L_1} + V_{0}^{LD} - V_{1}^{LD}, 0 \right\} + \frac{n_{1}^{SD}}{n_{D}} \max \left\{ -p_{L_1,S_1} + V_{2}^{LD} - V_{1}^{LD}, 0 \right\} + \frac{n_{2}^{LD}}{n_{D}} \max \left\{ -p_{L_1,L_1} + V_{2}^{LD} - V_{1}^{LD}, 0 \right\} \right\}. $$

36
An $L_2$, who may only sell, meets an investor-buyer at the rate $\eta(\theta_{ID})$. Among dealers, he may sell to an $S_0$ or an $L_0$. Then,

$$rV_{2LD} = \eta(\theta_{ID})(p_{L_2} + V_{1LD} - V_{2LD}) + \alpha \left\{ \frac{n_{0SD}}{n_D} \max\{p_{S_0L_2} + V_{1LD} - V_{2LD}, 0\} + \frac{n_{0LD}}{n_D} \max\{p_{L_0L_2} + V_{1LD} - V_{2LD}, 0\} \right\}. $$

**Investors** An investor-buyer may buy from an $S_1$, an $L_1$, or an $L_2$. Then,

$$rU^B = \mu(\theta_{ID}) \left( U_{H}^{ON} - U_{H}^{B} - \frac{n_{1SD}}{n_B}p_{H,S_1} - \frac{n_{1LD}}{n_B}p_{H,L_1} - \frac{n_{2LD}}{n_B}p_{H,L_2} \right), $$

where

$$rU_{H}^{ON} = \nu + \delta(U_{L}^{ON} - U_{H}^{ON}).$$

An investor-seller may sell to an $S_0$, an $L_0$, or an $L_1$. Then,

$$rU_{L}^{ON} = \eta(\theta_{DI}) \left( \frac{n_{1SD}}{n_B}p_{S_0L_s} + \frac{n_{0LD}}{n_B}p_{L_0L_s} + \frac{n_{1LD}}{n_B}p_{L_1L_s} - U_{L}^{ON} \right). $$

### 7.1.2 Equilibrium Conditions

To begin, since, by Proposition 3, among all dealers, $S_1$s only sell to $L_0$s whereas $S_0$s only buy from $L_2$s, in the steady state in which $n_{0SD}^* = n_{1SD}^* = 0$,

$$n_{1SD}^* \left( \eta(\theta_{ID}) + \alpha \frac{n_{1LD}^*}{n_D} \right) = n_{0SD}^* \left( \mu(\theta_{ID}) + \alpha \frac{n_{2LD}^*}{n_D} \right). \tag{38} $$

Also, where $L_1$s do not trade in the inter-dealer market and that $L_0$s buy from $S_1$s and $L_2$s, whereas $L_2$s sell to $S_0$s and $L_0$s, the equations for $n_{0LD}^* = 0$ and $n_{1LD}^* = 0$ specialize to, respectively,

$$n_{1LD}^* \eta(\theta_{ID}) = n_{0LD}^* \left( \mu(\theta_{ID}) + \alpha \frac{n_{1SD}^* + n_{2LD}^*}{n_D} \right), \tag{39} $$

$$n_{1LD}^* \mu(\theta_{DI}) = n_{2LD}^* \left( \eta(\theta_{ID}) + \alpha \frac{n_{0SD}^* + n_{1LD}^*}{n_D} \right). \tag{40} $$

Given $\{n_{SD}, n_{LD}, A, e, \delta\}$, a steady-state equilibrium consists of the respective non-negative values of $n_{0SD}^*, n_{1SD}^*, n_{0LD}^*, n_{1LD}^*, n_{2LD}^*, n_{0ON}^*, n_{L0}^*, n_{ON}^*$ and $n_{L}^*$ that satisfy the accounting identities in (22)-(24) and the steady-state conditions for $n_{0H}^*, n_{L}^*$ and $n_{L}^*$ in (27)-(29), which are common for both the competitive and frictional inter-dealer market models, and (38)-(40) above.

First, by (24),

$$n_{1SD}^* = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} - n_{1LD}^* - 2n_{2LD}^*, \tag{41} $$
from which it follows that
\[ n_1^{LD} + n_1^{SD} + n_2^{LD} = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} - n_2^{LD}. \]
Substitute the equation into (25) to yield
\[ n_2^{LD} = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} - \frac{e}{\eta(\theta_{ID})}. \] (42)
Next, by (22), (23), (41), and (42),
\[ n_0^{SD} + n_0^{LD} + n_1^{LD} = n^D - \frac{e}{\eta(\theta_{ID})} + n_1^{LD}. \]
Substitute the equation into (26) to yield
\[ n_1^{LD} = \frac{e}{\mu(\theta_{DI})} + \frac{e}{\eta(\theta_{ID})} - n^D. \] (43)
It is then straightforward to derive the following,
\[ n_0^{SD} = A - \frac{e}{\delta} - n^{LD} - \frac{e}{\eta(\theta_{DI})} - \frac{e}{\eta(\theta_{ID})} + \frac{e}{\mu(\theta_{DI})}, \] (44)
\[ n_1^{SD} = n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})} + \frac{e}{\eta(\theta_{ID})} - \frac{e}{\mu(\theta_{DI})}, \] (45)
\[ n_0^{LD} = n^{LD} + n^D - A + \frac{e}{\delta} - \frac{e}{\mu(\theta_{DI})} + \frac{e}{\eta(\theta_{ID})}. \] (46)
Any \( \{\theta_{ID}, \theta_{DI}\} \) pair satisfying (38) and (39) with the measures of dealers given by (41)-(46), where the resulting \( n_0^{SD}, n_1^{SD} \in [0, n^{SD}] \) and \( n_0^{LD}, n_1^{LD}, n_2^{LD} \in [0, n^{LD}] \), is a steady-state equilibrium.

7.2 Dealers’ Bid and Ask Prices

Lemma A1 In all three types of equilibrium, the dealers’ bid price; i.e., the price at which investors sell to dealers is given by
\[ p_{Is} = \frac{1 - \beta + \beta \eta(\theta_{DI})}{2(1 - \beta) + 3 \beta \eta(\theta_{DI})} p, \] (47)
whereas the dealers’ ask prices; i.e., the prices at which investors buy from dealers in the Selling, Balanced, and Buying Equilibria are given by, respectively,
\[ p_{IB} = \left(1 + \frac{1 - \beta}{3 \beta \eta(\theta_{ID})}\right) p, \] (48)
\[ p_{IB} = \left(1 + \frac{\left(1 - \beta + \beta \mu(\theta_{DI})\right)(4(1 - \beta) + 2 \beta \eta(\theta_{DI})) - \beta^2 \mu(\theta_{DI}) \eta(\theta_{DI})}{\beta \eta(\theta_{ID})(4(1 - \beta) + 3 \beta \theta_{DI})}\right) p, \] (49)
\[ p_{IB} = \frac{\beta(1 - \beta) \upsilon}{(1 - \beta + \beta \delta)(2(1 - \beta) + \mu(\theta_{1D})\beta)}. \]  

**Proposition A1** For \( e \leq n^{LD} \), as \( A \) increases from \( B_L + e/\delta \) at which the Selling Equilibrium first holds, \( p_{IS} \) and \( p_{IB} \) are continuously decreasing in \( A \). Once \( A \) reaches \( B_M + e/\delta \) at which the Balanced Equilibrium begins to hold, there will be discrete falls in the two prices. While the Balanced Equilibrium holds, \( p_{IS} \) is continuously decreasing in \( A \). And then finally, when \( A \) rises up to and above \( B_L + e/\delta \), at which the Buying Equilibrium holds, there will be further discrete falls in the two prices – \( p_{IS} \) all the way to zero and \( p_{IB} \) to some positive value. Thereafter, the two prices do not vary with \( A \) any longer. For \( e \in \left( n^{LD}, n^{LD} + \frac{n^{SD}}{2} \right) \) and that \( A > B_M + e/\delta \) at which the Balanced Equilibrium begins to hold, \( p_{IS} \) is likewise continuously decreasing in \( A \).

The Proposition leaves out how \( p_{IB} \) may vary with \( A \) in the Balanced Equilibrium as it does not seem possible to sign \( \partial p_{IB}/\partial A \) in said equilibrium. Our numerical analyses do reveal, however, that \( p_{IB} \) does decline in \( A \) while the Balanced Equilibrium holds,\(^{25}\) just as \( p \) and \( p_{IS} \) do.

For the smallest admissible \( n^{LD} \), the market starts off in a Balanced Equilibrium, in which \( p \), as our numerical analyses in the main text indicate, tends to decline with increases in \( n^{LD} \). In the same numerical analyses, we find that \( p_{IS} \) and \( p_{IB} \) follow the same tendency.

**Proposition A2** Holding fixed \( n^D \), as \( n^{LD} \) rises, if and when the Balanced Equilibrium gives way to the Buying Equilibrium, both \( p_{IS} \) and \( p_{IB} \) fall by some discrete amount – \( p_{IS} \) to zero and \( p_{IB} \) to some positive value; if and when the Balanced Equilibrium gives way to the Selling Equilibrium, both \( p_{IS} \) and \( p_{IB} \) jump up by some discrete amount. In either the Buying or the Selling Equilibrium, the two prices do not vary with \( n^{LD} \).

### 7.3 Efficient Decentralized Market Trades

A social planner maximizes the discounted flow payoffs for investors over time from the ownership of the asset given by,

\[ W = \max \left\{ \sum_{t=0}^{\infty} \beta^t n_H^{ON}(t) \upsilon \right\}, \tag{51} \]

subject to the same search and matching frictions that agents in the model face.

A priori, the equilibrium trades in the frictional investor-dealer market are constrained efficient where any trades with a positive surplus, but only such trades, will take place with the terms of trade in the bilateral meetings reached via Nash Bargaining. Specifically, any investor-buyer and dealer-seller trade is efficient with the former, but not the latter, deriving the flow payoff \( \upsilon \) in holding a unit of the asset. But then a dealer-seller becomes a dealer-seller

\(^{25}\)Under the same parameter configurations as for the numerical analyses preceding Proposition 4c, except that \( n^{LD} \) is fixed at 0.846.
in the first place only by acquiring the asset from an investor-seller. Then, any and all trades between an investor-seller and a dealer-buyer are also efficient.

This means that it suffices for us to ask how the planner may wish to allocate units of inventory among the dealers in each period after the investor-dealer trades are completed and whether the allocation coincides with the allocation that falls out from the inter-dealer market in equilibrium.

**Lemma A2** In the steady state of the planner’s solution, units of inventory not held by investors are allocated to dealers to maximize the measures of dealer-sellers (dealers who hold inventory) and dealer-buyers (dealers who possess spare capacity). To maximize the measure of dealer-sellers, first allocate one unit each to either small or large dealers, and then allocate any remaining inventory to the large dealers. To maximize the measure of dealer-buyers, first allocate one unit each to large dealers, and then allocate any remaining inventory to either large or small dealers. The two objectives are then attained simultaneously by allocating inventory in the following order: (1) one unit each to large dealers; (2) if there remains any inventory, then one unit each to small dealers; (3) if there remains any inventory, one more unit each to large dealers.

**Proposition A3** In the steady state, the allocations from the decentralized market trades coincide with the planning optimum.

### 7.4 Proofs of Lemmas and Propositions

**Proof of Lemma 1** Assuming equal bargaining power, the respective prices an investor-buyer pays to an $S_1$, an $L_1$, and an $L_2$ satisfy,

\[
\beta(U_{ON}^H - U^B) - p_{IB,S_1} = W_0^{SD} - W_1^{SD} + p_{IB,S_1},
\]

\[\beta(U_{ON}^H - U^B) - p_{IB,L_1} = W_0^{LD} - W_1^{LD} + p_{IB,L_1},\]  

\[\beta(U_{ON}^H - U^B) - p_{IB,L_2} = W_1^{LD} - W_2^{LD} + p_{IB,L_2}.\]

On the other hand, the respective prices an investor-seller receives from selling to an $S_0$, an $L_0$, and an $L_1$ satisfy,

\[p_{S_0,I_S} - \beta U_{ON}^{L_1} = W_1^{SD} - W_0^{SD} - p_{S_0,I_S},\]

\[p_{L_0,I_S} - \beta U_{ON}^{L_1} = W_1^{LD} - W_0^{LD} - p_{L_0,I_S},\]

\[p_{L_1,I_S} - \beta U_{ON}^{L_1} = W_2^{LD} - W_1^{LD} - p_{L_1,I_S}.\]

Notice that

\[W_1^{SD} = W_0^{SD} + p,\]

\[W_0^{LD} = W_1^{LD} - p,\]

\[W_2^{LD} = W_1^{LD} + p.\]

The lemma then follows from (52)-(57).
Proof of Proposition 1  Substitute (15) and (17) and (58)-(60) into the value functions (6) and (7) and (11)-(13),

\[ V_0^{SD} = W_0^{SD} + \frac{\mu(\theta_{DI})}{2} (p - \beta U_L^{ON}), \]  

\[ V_1^{SD} = W_0^{SD} + \left(1 - \frac{\eta(\theta_{ID})}{2}\right)p + \frac{\eta(\theta_{ID})}{2} \beta (U_H^{ON} - U_B), \]  

\[ V_0^{LD} = W_1^{LD} - \left(1 - \frac{\mu(\theta_{DI})}{2}\right)p - \frac{\mu(\theta_{DI})}{2} \beta U_L^{ON}, \]  

\[ V_1^{LD} = W_1^{LD} + \frac{\mu(\theta_{DI}) - \eta(\theta_{ID})}{2} p - \frac{\mu(\theta_{DI})}{2} \beta U_L^{ON} + \frac{\eta(\theta_{ID})}{2} \beta (U_H^{ON} - U_B), \]  

\[ V_2^{LD} = W_1^{LD} + \left(1 - \frac{\eta(\theta_{ID})}{2}\right)p + \frac{\eta(\theta_{ID})}{2} \beta (U_H^{ON} - U_B). \]  

We can then calculate

\[ (V_1^{LD} - V_0^{LD}) - (V_1^{SD} - V_0^{SD}) = \frac{\mu(\theta_{DI})}{2} (p - \beta U_L^{ON}), \]  

\[ (V_1^{SD} - V_0^{SD}) - (V_2^{LD} - V_1^{LD}) = \frac{\eta(\theta_{ID})}{2} (\beta (U_H^{ON} - U_B) - p). \]  

Notice that the terms inside the brackets in (66) and (67) denote, respectively, the surpluses of trade between an investor-seller and any dealer-buyer and between an investor-buyer and any dealer-seller in (14) and (16). If either of the two is negative, there cannot be any trade in equilibrium between investors and dealers in the steady state.

Proof of Lemma 2  Substitute (15) into (3) and rearrange,

\[ U_L^{ON} = \frac{\eta(\theta_{DI})}{1 - \beta + \beta \frac{\mu(\theta_{DI})}{2}} p. \]  

Substitute the equation into (2) and rearrange,

\[ U_H^{ON} = \frac{(1 - \beta + \frac{\eta(\theta_{DI})}{2} \beta) v + \beta \delta \frac{\eta(\theta_{DI})}{2} p}{(1 - \beta + \beta \delta) \left(1 - \beta + \frac{\mu(\theta_{DI})}{2} \beta\right)}. \]  

Substitute (17) into (1) and rearrange,

\[ U_B = \frac{\mu(\theta_{ID})}{2} \beta \frac{U_H^{ON} - p}{1 - \beta + \beta \frac{\mu(\theta_{DI})}{2}}. \]  

Substituting from (69),

\[ U_B = \frac{\mu(\theta_{ID})}{2} \beta \frac{(1 - \beta + \frac{\eta(\theta_{DI})}{2} \beta) v - (1 - \beta) \left(1 - \beta + \beta \delta + \frac{\eta(\theta_{DI})}{2} \beta\right) p}{(1 - \beta + \beta \delta) \left(1 - \beta + \beta \frac{\mu(\theta_{DI})}{2} \beta\right)}. \]
Then, by (69) and (71),
\[ U_H^{ON} - U_B = \frac{\left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) + \beta \theta \frac{\mu(\theta D)}{2} \left( 1 - \beta \right) + \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta \right) \nu}{(1 - \beta + \beta \theta \frac{\mu(\theta D)}{2}) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta \right)} \]  
(72)

Set \( p = \beta \left( V_1^{LD} - V_0^{LD} \right) \) and by (63) and (64),
\[ p = \frac{\beta \eta(\theta D)}{1 - \beta + \beta \theta \frac{\mu(\theta D)}{2}} \beta \left( U_H^{ON} - U_B \right). \]  
(73)

Then use (72) to obtain
\[ p = \frac{\beta^2 \theta \frac{\mu(\theta D)}{2} \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right)^2}{(1 - \beta + \beta \theta \frac{\mu(\theta D)}{2}) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta \right)}. \]  
(74)

Given the positivity of \( p \) in (74) and by (68) and (73),
\[ 0 < \beta U_L^{ON} < p < \beta \left( U_H^{ON} - U_B \right). \]

Next, set \( p = \beta \left( V_1^{SD} - V_0^{SD} \right) \) and by (61) and (62),
\[ p = \frac{\eta(\theta D) \beta^2 \left( U_H^{ON} - U_B \right) + \theta \frac{\mu(\theta D)}{2} \beta^2 U_L^{ON}}{1 - \beta + \beta \left( \theta \frac{\mu(\theta D)}{2} + \theta \frac{\mu(\theta D)}{2} \right)}. \]  
(75)

Then use (68) and (72) to obtain
\[ p = \frac{\eta(\theta D) \beta^2 \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right)^2}{\left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta \right)}. \]  
(76)

Given the positivity of \( p \) in (76) and by (68) and (75),
\[ 0 < \beta U_L^{ON} < p < \beta \left( U_H^{ON} - U_B \right). \]

Finally, set \( p = \beta \left( V_2^{LD} - V_1^{LD} \right) \) and by (64) and (65),
\[ p = \frac{\beta \mu(\theta D) \beta}{1 - \beta + \beta \theta \frac{\mu(\theta D)}{2}} \beta U_L^{ON}. \]  
(77)

With (68),
\[ p = \beta U_L^{ON} = 0. \]

Next, by (72),
\[ U_H^{ON} - U_B = \frac{(1 - \beta) \nu}{(1 - \beta + \beta \theta \frac{\mu(\theta D)}{2}) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta + \beta \theta \frac{\mu(\theta D)}{2} \right) \left( 1 - \beta \right)} > 0. \]  
(78)

Thus,
\[ 0 = \beta U_L^{ON} = p < \beta \left( U_H^{ON} - U_B \right). \]
Proof of Proposition 2  Before proceeding to prove the Proposition, it is useful to establish the following.

Remark 1  For $x \leq 1$, $x < \eta^{-1}(x)$.
Proof. Given $\mu(x) = \eta(x) / x$ and that $\mu(x) < 1$, $\eta(x) < x$. And then for $x \leq 1$, the last condition implies $x < \eta^{-1}(x)$.

Remark 2  For $x \geq 1$, $x > \mu^{-1}\left(\frac{1}{2}\right)$.
Proof. Given that $\eta(x) = x\mu(x)$ and that $\eta(x) < 1$, $\mu(x) < \frac{1}{2}$. And then for $x \geq 1$, the last condition implies $x > \mu^{-1}\left(\frac{1}{2}\right)$.

Remark 3  For $\delta \leq n$, $\frac{n}{\mu^{-1}(\frac{1}{2})}$ is decreasing in $n$.
Proof. By differentiation.

To begin proving the Proposition, we start with manipulating (27)-(29) to obtain,

\[ n_{ON}^H = \frac{e}{\delta}, \]  \hspace{1cm} (79)

\[ n_{ON}^L = \frac{e}{\eta(\theta_{DI})}, \]  \hspace{1cm} (80)

\[ n_{B} = \frac{e}{\mu(\theta_{ID})}. \]  \hspace{1cm} (81)

Selling Equilibrium  In the Selling Equilibrium, $n_{2}^{LD} = n_{1}^{SD} = 0$ and $n_{0}^{SD} = n^{SD}$. Then, together with (80) and (81), the two market tightness equations, (25) and (26), specialize to, respectively,

\[ \eta(\theta_{ID}) = \frac{e}{n_{1}^{LD}}, \]  \hspace{1cm} (82)

\[ \mu(\theta_{DI}) = \frac{e}{n_{D}}. \]  \hspace{1cm} (83)

By (24), (79), (80), and that $n_{1}^{SD} = n_{2}^{LD} = 0$ in the Selling Equilibrium,

\[ n_{1}^{LD} = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})}. \]  \hspace{1cm} (84)

Substitute the equation into (82) and rearrange,

\[ \eta(\theta_{ID}) = \frac{\eta(\theta_{DI})e}{(A - \frac{e}{\delta})\eta(\theta_{DI})} - e. \]  \hspace{1cm} (85)

Once $\theta_{DI}$ is known from (83), the above uniquely gives $\theta_{ID}$. For $\theta_{DI}$ from (83) to be a valid equilibrium, it has to be such that the resulting: (a) $\eta(\theta_{ID}) \in (0, 1)$, as given by (85) and (b) $n_{1}^{LD} \in (e, n_{1}^{LD})$, as given by (84). For (b) to be satisfied, $e < n^{LD}$ must hold.

By (85), for $\eta(\theta_{ID}) \in (0, 1)$,

\[ \frac{e}{\eta(\theta_{DI})} < A - \frac{e}{\delta} - e. \]  \hspace{1cm} (RS.1)
Substituting from (83) and rearranging, (RS.1) holds if

\[ A - \frac{e}{\delta} > e + \frac{n^D}{\mu^{-1}(\frac{e}{n^D})} = B_S. \]  

Given that \( \eta(\theta_{ID}) < 1 \), \( n_1^{LD} > e \) holds for sure. By (84), for \( n_1^{LD} \leq n^{LD} \),

\[ \frac{e}{\eta(\theta_{DI})} \geq A - \frac{e}{\delta} - n^{LD}. \]  

(RS.2)

Given \( \theta_{DI} \) from (83), the above becomes,

\[ A - \frac{e}{\delta} \leq n^{LD} + \frac{n^D}{\mu^{-1}(\frac{e}{n^D})} = B_M. \]  

(AS.2)

Notice that for (RS.1) and (RS.2) to be satisfied at the same time, it has to be such that \( e < n^{LD} \), in which case (83) and the two conditions (AS.1) and (AS.2) are guaranteed to be well-defined.

Substituting (31), (32), and (34) into the inter-dealer market equilibrium condition (18),

\[ (n^{SD} + n_1^{LD}) \mu(\theta_{DI}) \leq n_0^{DD} (1 - \mu(\theta_{DI})) + n_1^{LD} \eta(\theta_{ID}). \]

And then with (82)-(85), the condition can be shown to simplify to (RS.2).

To sum up, the Selling Equilibrium holds if and only if (AS.1) and (AS.2) hold; i.e., \( A - \frac{e}{\delta} \in (B_S, B_M] \), in addition to \( e < n^{LD} \).

**Buying Equilibrium** In the Buying Equilibrium, \( n_0^{LD} = n_0^{SD} = 0 \) and \( n_1^{SD} = n^{SD} \). Then, together with (80) and (81), the two market tightness equations, (25) and (26), specialize to, respectively,\n
\[ \eta(\theta_{ID}) = \frac{e}{n^{DD}}, \]  

(87)

\[ \mu(\theta_{DI}) = \frac{e}{n_1^{DD}}. \]  

(88)

By (24), (79), (80), and that \( n_0^{SD} = n_0^{LD} = 0 \) in the Buying Equilibrium,

\[ n_1^{LD} = n^{SD} + 2n^{LD} - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}. \]  

(89)

Substitute the equation into (88),

\[ e(\theta_{DI} - 1) - \eta(\theta_{DI}) \left( n^{SD} + 2n^{LD} - A + \frac{e}{\delta} \right) = 0, \]  

(90)

which is an equation in \( \theta_{DI} \) alone. It is straightforward to verify that there is a unique positive solution of \( \theta_{DI} \) to the equation, and that the LHS is positive (negative) for \( \theta_{DI} \) above (below) the solution of the equation. For the solution to be a valid equilibrium, it has to be such that the resulting \( n_1^{LD} \in (e, n^{LD}] \), as given by (89). Then \( e < n^{LD} \) must be satisfied.
Rearranging (89), \( n_{1}^{LD} \leq n^{LD} \) if and only if
\[
\frac{e}{\eta(\theta_{DI})} \leq A - \frac{e}{\delta} - n^{D}.
\] (RB.1)

A necessary condition for the equation to hold is that
\[
A - \frac{e}{\delta} - n^{D} \geq e.
\] (AB1.a)

Then, (RB.1) holds if the LHS of (90) is non-positive when evaluated at \( \theta_{DI} = \eta^{-1}\left(\frac{e}{A - \frac{e}{\delta} - n^{D}}\right) \).

Where \( e < n^{LD} \), which is necessary for the RHS of (88) and also guarantees the RHS of (87) to be bounded below one, the condition reads
\[
A - \frac{e}{\delta} \geq n^{D} + \frac{n^{LD}}{\mu^{-1}\left(\frac{e}{n^{LD}}\right)} = BL,
\] (AB1.b)

which subsumes (AB1.a).

Rearranging (89), \( n_{1}^{LD} > e \) if and only if
\[
A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD} < \frac{e}{\eta(\theta_{DI})}.
\] (RB.2)

Hence, if
\[
A - \frac{e}{\delta} - n^{SD} - 2n^{LD} \leq 0,
\]
then (RB.2) holds for sure. Otherwise, the condition holds if the LHS of (90) is positive when evaluated at \( \theta_{DI} = \eta^{-1}\left(\frac{e}{A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD}}\right) \); i.e.,
\[
\eta^{-1}\left(\frac{e}{A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD}}\right) > \frac{e}{A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD}}.
\]

But the condition is guaranteed to hold by Remark 1.

Substituting (30), (32), and (34) into the inter-dealer market equilibrium condition (21),
\[
(n^{SD} + n_{1}^{LD}) \eta(\theta_{ID}) \leq n_{1}^{LD} \mu(\theta_{DI}) + n_{2}^{LD} (1 - \eta(\theta_{ID})).
\]

Then, by (87)-(90), the condition can be shown to simplify to (RB.1).

To sum up, the Buying Equilibrium holds if and only if (AB1.b) holds; i.e., \( A - \frac{e}{\delta} \geq BL \), in addition to \( e < n^{LD} \).

Balanced Equilibrium In the Balanced Equilibrium, \( n_{0}^{ID} = n_{2}^{ID} = 0 \) and \( n_{1}^{ID} = n^{LD} \).

Then, together with (80) and (81), the two market tightness equations, (25) and (26), specialize to, respectively,
\[
\eta(\theta_{ID}) = \frac{e}{n_{1}^{SD} + n^{LD}},
\] (92)
\[
\mu(\theta_{DI}) = \frac{e}{n_{0}^{SD} + n^{LD}}.
\] (93)
By (24), (79), and (80), and that $n_0^{LD} = n_2^{LD} = 0$ in the Balanced Equilibrium,

$$n_1^{SD} = A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{ID})},$$

and therefore

$$n_0^{SD} = n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{ID})}.$$  \hspace{1cm} (95)

Substituting (94) and (95) into (92) and (93), respectively, and rearranging,

$$\eta(\theta_{ID}) = \frac{\eta(\theta_{DI}) e}{(A - e/\delta) \eta(\theta_{DI}) - e},$$

$$\epsilon(\theta_{DI} - 1) - \eta(\theta_{DI}) \left( n^D + n^{LD} - A + \frac{e}{\delta} \right) = 0,$$

which are respectively the same equations that give $\theta_{ID}$ in the Selling Equilibrium in (85) and $\theta_{DI}$ in the Buying Equilibrium in (90).

For now, we restrict attention to where $\epsilon < \frac{n^D + n^{LD}}{2}$. Later on, we will verify that the condition is necessary for the existence of the Balanced Equilibrium. For the solution of (97) to be a valid equilibrium, it has to be such that (a) $n_0^{SD} = n_0^{SD} \in [0, n^{SD}]$ for $\epsilon < n^{LD}$ and $n_0^{SD} \in (n^{LD}, n^D - \epsilon)$ for $\epsilon \in [n^{LD}, \frac{n^D + n^{LD}}{2}]$ and (b) $\eta(\theta_{ID}) \in (0, 1)$, as given by (96).

By (95), where $\epsilon < n^{LD}$, for $n_0^{SD} \geq 0$,

$$\frac{e}{\eta(\theta_{DI})} \geq A - \frac{e}{\delta} - n^D$$

has to hold. The condition is guaranteed to hold if

$$A - \frac{e}{\delta} - n^D \leq \epsilon.$$ \hspace{1cm} (ABA.1a)

Otherwise, (ABA.1) can only hold if the LHS of (97) is non-negative when evaluated at $\theta_{DI} = \eta^{-1} \left( A - \frac{e}{\delta} - n^D \right)$; i.e.,

$$A - \frac{e}{\delta} \leq n^D + \frac{n^{LD}}{\mu^{-1} \left( \frac{e}{\delta - n^D} \right)} = B_L.$$ \hspace{1cm} (ABA.1b)

Note that (ABA.1) holds if either (ABA.1a) or (ABA.1b) is satisfied. Given that $\frac{n^{LD}}{\mu^{-1} \left( \frac{e}{\delta - n^D} \right)} < \epsilon$ by Remark 2, however, the latter condition subsumes the former one to begin with. Next, for $n_0^{SD} \leq n^{SD}$, by (95),

$$\frac{e}{\eta(\theta_{DI})} \leq A - \frac{e}{\delta} - n^{LD}.$$ \hspace{1cm} (RBA.2)

The condition holds if the LHS of (97) is non-positive when evaluated at $\theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^D} \right)$; i.e.,

$$A - \frac{e}{\delta} \geq n^{LD} + \frac{n^D}{\mu^{-1} \left( \frac{e}{\delta - n^D} \right)} = B_M.$$ \hspace{1cm} (ABA.2)
Where \( e \in \left[ n^{LD}, \frac{n^{D} + n^{LD}}{2} \right] \), for \( n_0^{SD} > e - n^{LD} \),
\[
\frac{e}{\eta(\theta_{DI})} > A - \frac{e}{\delta} - n^{D} - n^{LD} + e
\] (RBA.3)
has to hold. The condition is guaranteed to hold if
\[
A - \frac{e}{\delta} - n^{D} - n^{LD} \leq 0.
\]
Otherwise, (RBA.3) can only hold if the LHS of (97) is positive when evaluated at \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{D} - n^{LD} + e} \right) \); i.e.,
\[
\eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{D} - n^{LD} + e} \right) - \frac{e}{A - \frac{e}{\delta} - n^{D} - n^{LD} + e} > 0.
\]
This inequality is met for sure by Remark 1, as \( A - \frac{e}{\delta} - n^{D} - n^{LD} > 0 \) implies \( \frac{e}{A - \frac{e}{\delta} - n^{D} - n^{LD} + e} < 1 \). Next, for \( n_0^{SD} < n^{D} - e \), by (95),
\[
\frac{e}{\eta(\theta_{DI})} < A - \frac{e}{\delta} - e,
\] (RBA.4)
By (96), the condition for \( \eta(\theta_{ID}) \in (0, 1) \) is the same condition as (RBA.4). The condition holds if the LHS of (97) is negative when evaluated at \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right) \); i.e.,
\[
\frac{n^{D} + n^{LD} - e}{A - \frac{e}{\delta} - e} - \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right) > 0.
\] (98)
The condition can only hold for \( e < \frac{n^{D} + n^{LD}}{2} \), justifying our previous claim that the Balanced Equilibrium can only hold for \( e \) bounded from below the given value, because, otherwise,
\[
\frac{n^{D} + n^{LD} - e}{A - \frac{e}{\delta} - e} \leq \frac{e}{A - \frac{e}{\delta} - e} < \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right),
\]
where the last inequality is by Remark 1. Given \( e < \frac{n^{D} + n^{LD}}{2} \), (98) is equivalent to
\[
A - \frac{e}{\delta} > e + \frac{n^{D} + n^{LD} - e}{\mu^{-1} \left( \frac{e}{n^{D} + n^{LD} - e} \right)} = \mathcal{B}_M.
\] (ABA.4)

The condition for there to be more sellers than buyers among large dealers in the inter-dealer market \( m_2^{LD} \geq m_0^{LD} \), by (32) and (34), and \( n_0^{LD} = n_2^{LD} = 0 \) in the Balanced Equilibrium, becomes
\[
\mu(\theta_{DI}) \geq \eta(\theta_{ID}).
\]
By (96),
\[
\mu(\theta_{DI}) - \eta(\theta_{ID}) = \mu(\theta_{DI}) \frac{(A - \frac{e}{\delta}) \eta(\theta_{DI}) - e \theta_{DI}}{(A - \frac{e}{\delta}) \eta(\theta_{DI}) - e}.
\]
The denominator is guaranteed positive for $\eta(\theta_{ID}) \in [0, 1]$. The expression then has the same sign as the numerator; i.e.,

$$\mu(\theta_{DI}) - \eta(\theta_{ID}) \geq 0 \Leftrightarrow \eta(\theta_{DI}) \left( A - \frac{e}{\delta} \right) - e - e\theta_{DI} \geq 0.$$ 

Rewrite (97) as

$$\eta(\theta_{DI}) \left( A - \frac{e}{\delta} \right) - e - e\theta_{DI} = \eta(\theta_{DI}) \left( n^D + n^{LD} \right) - 2e\theta_{DI}. \quad (100)$$

We seek conditions on how the two sides of the equation meet at a non-negative value.

Properties of $\eta(\theta) \left( A - \frac{e}{\delta} \right) - e - e\theta$

1. equal to $-e$ at $\theta = 0$,
2. tends to negative infinity as $\theta \to \infty$,
3. given condition (ABA.4), so that $A - \frac{e}{\delta} - e > 0$, is inverted-U,
4. if $\max \{ \eta(\theta) \left( A - \frac{e}{\delta} \right) - e - e\theta \} > 0$, rises above zero for a range of $\theta$.

Properties of $\eta(\theta) \left( n^D + n^{LD} \right) - 2e\theta$

1. equal to 0 at $\theta = 0$,
2. tends to negative infinity as $\theta \to \infty$.
3. For $e < \frac{n^D + n^{LD}}{2}$, is inverted-U.

Given these properties of the two sides of (100), the RHS is greater than the LHS before the two sides meet, whereas the LHS is less than the RHS thereafter. Then, if at where the RHS vanishes, i.e.,

$$\theta = \mu^{-1} \left( \frac{2e}{n^D + n^{LD}} \right),$$

the LHS is non-negative; i.e.,

$$A - \frac{e}{\delta} \geq \frac{n^D + n^{LD}}{2} + \frac{n^D + n^{LD}}{2} \mu^{-1} \left( \frac{2e}{n^D + n^{LD}} \right) = S, \quad (ABA.S)$$

then the meeting point is where the two sides are non-negative.

If (ABA.S) holds, the relevant inter-dealer market equilibrium condition is (19), which becomes

$$n^{LD} (\mu(\theta_{DI}) - \eta(\theta_{ID})) \leq \left( n^D_{1D} (1 - \mu(\theta_{DI})) + n^D_{1D} \eta(\theta_{ID}) \right),$$

after substituting in (30), (32), and (34). By (94)-(96), the condition becomes

$$\left( n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})} \right) (1 - \mu(\theta_{DI})) + \left( \frac{e}{\mu(\theta_{DI})} - n^{LD} \right) \mu(\theta_{DI}) \geq 0. \quad (RBA.5)$$

Rewrite (97) as

$$\frac{e}{\mu(\theta_{DI})} - n^{LD} = n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}$$
The LHS of (RBA.5) is a weighted average of the two terms in this equation. Thus, if the equation holds where the two sides are non-negative, (RBA.5) must hold. In turn, in case \( e < n^{LD} \) and if (RBA.1) holds, under which \( n_0^{SD} \geq 0 \), and in case \( e \in [n^{LD}, \frac{n^{DF}+n^{LD}}{2}] \) and if (RBA.3) holds, under which \( n_0^{SD} > e - n^{LD} \), the RHS of the equation is guaranteed non-negative.

If (ABA.S) holds in reverse, the relevant inter-dealer market equilibrium condition is (20), which becomes

\[
n^{LD} (\eta(\theta_{ID}) - \mu(\theta_{DI})) \leq n_0^{SD} \mu(\theta_{DI}) + n_1^{SD} (1 - \eta(\theta_{ID})),
\]

after substituting in (31), (32), and (34). By (94)-(96), the condition becomes

\[
(A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})}) (1 - \mu(\theta_{DI})) + (n^{D} - \frac{e}{\mu(\theta_{DI})}) \mu(\theta_{DI}) \geq 0.
\]

Rewrite (97) as

\[
A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} = n^{D} - \frac{e}{\mu(\theta_{DI})}.
\]

The LHS of (RBA.6) is a weighted average of the two terms in this equation. Thus if the equation holds at the point where the two sides are non-negative, (RBA.6) must hold. In turn, in case \( e < n^{LD} \) and if (RBA.2) holds, under which \( n_0^{SD} \leq n^{SD} \) and in case \( e \in [n^{LD}, \frac{n^{DF}+n^{LD}}{2}] \) and if (RBA.4) holds, under which \( n_0^{SD} < n^{D} - e \), the LHS of the equation is guaranteed non-negative.

Notice that in case \( e < n^{LD} \), (RBA.2) is a more stringent condition than (RBA.4). Then, for \( \theta_{ID} \) and \( \theta_{DI} \) defined by (96) and (97) to be a valid Balanced Equilibrium, it suffices that (ABA.1b) and (ABA.2) hold; i.e., \( A - \frac{e}{\delta} \in [B_M, B_L] \). Otherwise for \( e \in [n^{LD}, \frac{n^{DF}+n^{LD}}{2}] \), the equilibrium holds under (ABA.4); i.e., \( A - \frac{e}{\delta} > B_M \). In either case, for \( A - \frac{e}{\delta} \leq S \), small dealers sell in equilibrium; otherwise small dealers buy.

**Ranking of the Bounds**  That \( B_S \leq B_M \) follows from \( e \leq n^{LD} \), whereas that \( B_M \leq S \leq B_L \) follows from \( n^{LD} \leq n^{D} \) and Remark 3. That \( B_M \leq S \) follows from \( e \leq n^{LD} + \frac{n^{SD}}{2} \) and Remark 3.

**Proof of Corollary 1**  The condition \( e \in [n^{LD}, n^{LD} + \frac{n^{SD}}{2}] \) is equivalent to \( n^{LD} \in (2e - n^{D}, e] \). For such \( n^{LD} \), the Balanced Equilibrium indeed holds if the condition in Proposition 2(b) is met, which can be rewritten as the first condition of the Proposition. Notice that the RHS of the condition is greater than \( 2e - n^{D} \) by Remark 1 in the proof of Proposition 2, meaning that any \( n^{LD} \) that satisfies the condition exceeds \( 2e - n^{D} \). Now when \( n^{LD} \) rises up to \( e \), Proposition 2(a) applies. At \( n^{LD} = e \), \( B_L \rightarrow \infty \) and \( B_M = B_S \), in which case the Balanced Equilibrium continues to hold. At \( n^{LD} = n^{D} \),

\[
B_M = S = B_L = \mathcal{T},
\]

in which case the Balanced Equilibrium still holds only if \( A - e/\delta = \mathcal{T} \). Otherwise, for \( A - e/\delta < (>) \mathcal{T} \), the Selling (Buying) Equilibrium holds. In general, as \( n^{LD} \) increases from
e to \( n^D \), \( B_L \) falls from infinity, whereas \( B_M \) and \( S \) go up and diverge from \( B_S \). Eventually the three bounds converge to \( \bar{B} \). Then, for \( A - e/\delta < (>) \bar{B} \), the Balanced Equilibrium must turn into the Selling (Buying) Equilibrium at some \( n^{LD} \in (e, n^D) \). The cutoff values are from Proposition 2(a).

**Proof of Corollary 2** For \( e < n^{LD} \), in both the Selling Equilibrium, which holds for \( A - e/\delta \in (B_S, B_M) \), and in the Balanced Equilibrium for \( A - e/\delta \in [B_M, S] \), small dealers sell to large dealers. In both the Balanced Equilibrium for \( A - e/\delta \in (S, B_L) \) and the Buying Equilibrium, which holds for \( A - e/\delta \geq B_L \), small dealers buy from large dealers. For \( e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right) \), only the Balanced Equilibrium can hold, in which the cutoff value for \( A - e/\delta \) is similarly \( S \).

**Proof of Lemma 3** First, that all dealers are dealer-buyers and dealer-sellers and that \( \alpha e \) measures of small investors buy and sell in each period imply that

\[
\alpha e = n^D \eta(\theta_{ID}) \alpha, \\
\alpha e = n^D \mu(\theta_{DI}) \alpha.
\]

The two conditions combine to yield \( \eta(\theta_{ID}) = \mu(\theta_{DI}) \), which in turn implies that \( m_0^{SD} = m_2^{SD} \), given that

\[
m_0^{SD} = \eta(\theta_{ID}) \alpha (1 - \mu(\theta_{DI}) \alpha) n^{SD}, \\
m_2^{SD} = \mu(\theta_{DI}) \alpha (1 - \eta(\theta_{ID}) \alpha) n^{SD}.
\]

Next, with \((1 - \alpha) e\) measure of large investors buying from large dealers holding a two-unit inventory as well as \((1 - \alpha) e\) measure of large investors selling to large dealers holding a one-unit inventory,

\[
(1 - \alpha) e = n_2^{LD} \eta(\theta_{ID}) (1 - \alpha), \\
(1 - \alpha) e = n_1^{LD} \mu(\theta_{DI}) (1 - \alpha).
\]

Given that \( \eta(\theta_{ID}) = \mu(\theta_{DI}) \), the above imply \( n_1^{LD} = n_2^{LD} = n^{LD}/2 \). To follow then is \( m_0^{LD} = m_3^{LD} \) given that

\[
m_0^{LD} = n_1^{LD} \eta(\theta_{ID}) \alpha (1 - \mu(\theta_{DI})) + n_2^{LD} \eta(\theta_{ID}) (1 - \alpha) (1 - \alpha \mu(\theta_{DI})), \\
m_3^{LD} = n_1^{LD} \mu(\theta_{DI}) (1 - \alpha) (1 - \alpha \eta(\theta_{ID})) + n_2^{LD} \mu(\theta_{DI}) \alpha (1 - \eta(\theta_{ID})).
\]

**Proof of Lemma 4** The surpluses of the possible trades are as follows.

\[
z_{B,S_i} = U_i^{ON} - U_B - (V_i^{SD} - V_0^{SD}), \\
z_{B,L_i} = U_i^{ON} - U_B - (V_i^{LD} - V_{i-1}^{LD}) \text{ for } i = 1, 2, \\
z_{S_o,L} = V_1^{SD} - V_0^{SD} - U_L^{ON}, \\
z_{L_i,S} = V_i^{LD} - V_{i+1}^{LD} - U_L^{ON} \text{ for } i = 0, 1,
\]

50
For $i = 1, 2$:

$$z_{S_0,L_i} = V_i^{SD} - V_0^{SD} - (V_i^{LD} - V_{i-1}^{LD})$$

For $i = 0, 1$:

$$z_{L_0,S_1} = V_{i+1}^{LD} - V_i^{LD} - (V_1^{SD} - V_0^{SD})$$

$$z_{L_0,L_2} = V_1^{LD} - V_0^{LD} - (V_2^{LD} - V_1^{LD})$$

$$z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - (V_1^{LD} - V_0^{LD})$$

An $S_1$ can sell to an IB (investor-buyer), an $L_0$, or an $L_1$. He will not sell to an IB only if selling to other dealers yields a strictly larger surplus; i.e.,

$$\max \{z_{L_0,S_1}, z_{L_1,S_1}\} > z_{IB,S_1}.$$  

Expanding the expressions for the $z_i$,

$$\max \{V_1^{LD} - V_0^{LD}, V_2^{LD} - V_1^{LD}\} > U_H^{ON} - U^B.$$  

Subtracting $U_H^{ON} - U^{IB}$ from the two sides of the condition

$$\max \{V_1^{LD} - V_0^{LD} - (U_H^{ON} - U^B), V_2^{LD} - V_1^{LD} - (U_H^{ON} - U^B)\} > 0.$$  

The two terms inside the max operator are simply the negatives of $z_{IB,L_1}$ and $z_{IB,L_2}$, respectively. Then, the condition becomes

$$\max \{-z_{IB,L_1}, -z_{IB,L_2}\} > 0 \Leftrightarrow \min \{z_{IB,L_1}, z_{IB,L_2}\} < 0.$$  

All this implies that if one type of dealer-seller finds it optimal not to sell to investor-buyers, then only one type of dealer-seller may find it optimal to do so. In any active steady-state equilibrium, indeed at least one type of dealer-seller must do so.

Now, suppose only $S_1$s sell to IB where

$$z_{IB,S_1} = U_H^{ON} - U^B - (V_1^{SD} - V_0^{SD}) \geq 0. \tag{102}$$

An $L_1$ may then only sell to an $S_0$ or another $L_1$. Selling to an $S_0$ is optimal if

$$z_{S_0,L_1} = V_1^{SD} - V_0^{SD} - (V_1^{LD} - V_0^{LD}) \geq 0.$$  

But if the condition holds,

$$z_{IB,L_1} = U_H^{ON} - U^B - (V_1^{LD} - V_0^{LD}) \geq 0$$

must hold given (102). The hypothesis that only $S_1$ sell to IB then implies that selling to another $L_1$ must be optimal for the $L_1$ (otherwise the $L_1$ has no one to sell to), where

$$z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - (V_1^{LD} - V_0^{LD}) \geq 0. \tag{103}$$

An $L_2$ may sell to an $S_0$ or an $L_0$ if selling to an IB is not optimal. Selling to an $S_0$ is optimal if

$$z_{S_0,L_2} = V_1^{SD} - V_0^{SD} - (V_2^{LD} - V_1^{LD}) \geq 0.$$  

51
But if the condition holds,

\[ z_{Ib, L_2} = U^Q_{Ib} - U^B - (V^L_2 - V^L_1) \geq 0 \]

must hold given (102). The hypothesis that only \( S_1 \) sell to \( IB \) then implies that selling to an \( L_0 \) must be optimal for the \( L_2 \), where

\[ z_{L_0, L_2} = V^L_1 - V^L_0 - (V^L_2 - V^L_1) \geq 0. \quad (104) \]

The two conditions, (103) and (104), together imply that

\[ V^L_1 - V^L_0 = V^L_2 - V^L_1. \]

Thus, if neither \( L_1 \) nor \( L_2 \) find it optimal to sell to investor-buyers or to small dealers, large dealers do not gain by selling and buying among themselves either. They must then be inactive in equilibrium.

Next, suppose only \( L_1 \) sell to investor-buyers, where

\[ z_{Ib, L_1} = U^Q_{Ib} - U^B - (V^L_2 - V^L_1) \geq 0. \quad (105) \]

An \( S_1 \) may sell to an \( L_0 \) or to an \( L_1 \) if not selling to an investor-buyer. If the first sale is optimal, it must be optimal for the \( S_1 \) to sell to an \( IB \) as well given (105). The hypothesis that only \( L_1 \) sells to investor-buyers then requires that it is optimal for an \( S_1 \) to sell to an \( L_1 \) where

\[ z_{L_1, S_1} = V^L_2 - V^L_0 - (V^S_1 - V^S_0) \geq 0. \quad (106) \]

An \( L_2 \) may sell to an \( L_0 \) or to an \( S_0 \). If the first sale is optimal, it must be optimal for the \( L_2 \) to sell to an \( IB \) as well given (105). The condition for the second sale to be optimal is that

\[ z_{S_0, L_2} = V^S_1 - V^S_0 - (V^L_2 - V^L_1) \geq 0. \quad (107) \]

The two conditions, (106) and (107), together imply that

\[ V^S_1 - V^S_0 = V^L_2 - V^L_1. \]

Thus, if neither \( S_1 \) nor \( L_2 \) find it optimal to sell to investor-buyers, \( S_1 \) only sell to \( L_1 \), where such trades do not yield any surplus. This implies that small dealers must be inactive in equilibrium.

The case for where only \( L_2 \) sell to investor-buyers can be shown in a similar way to imply that small dealers must be inactive in equilibrium.

The proof that in any equilibrium in which both small and large dealers are active, investor-sellers must sell to all three types of dealer-buyers can be constructed similarly.

**Proof of Proposition 3**  With Nash Bargaining and each agent in a match entitled to one-half of the match’s surplus, we can rewrite dealers’ value functions as follows.

\[ rV^S_0 = \mu(\theta_{DI}) \frac{n^L_{Ib}I_{Ib}}{2} + \alpha \left\{ \frac{n^L_{Ib}}{2n^I_{Ib}} \max\{z_{S_0, L_1}, 0\} + \frac{n^L_{Ib}}{2n^I_{Ib}} \max\{z_{S_0, L_2}, 0\} \right\}, \]

52
Proof of Proposition 4a In the Selling Equilibrium, \( \theta_{DI} \) is implicitly given by (83), in which \( A \) is absent. By (85), \( \theta_{ID} \) is decreasing in \( A \) given that \( \theta_{DI} \) does not vary with \( A \).

In the Balanced Equilibrium, \( \theta_{DI} \) is implicitly given by (97), the solution to which is at a point where the LHS of the equation is increasing. In the meantime, the LHS of the equation is increasing in \( A \). Then, \( \partial \theta_{DI} / \partial A < 0 \). To evaluate the effect of \( A \) on \( \theta_{ID} \), first rewrite (97) as

\[
A - \frac{e}{\delta} = n^D + n^{LD} - \frac{e (\theta_{DI} - 1)}{\eta (\theta_{DI})}.
\]
Then substitute the equation into (96) to yield

\[ \eta(\theta_{ID}) = \frac{\eta(\theta_{DI}) e}{(n^D + n^{LD}) \eta(\theta_{DI}) - e\theta_{DI}}, \]

the RHS of which is increasing in \( \theta_{DI} \) due to the concavity of \( \eta \). Then, \( \theta_{ID} \) must be decreasing in \( A \).

In the Buying Equilibrium, \( \theta_{ID} \) is implicitly given by (87), in which \( A \) is absent, whereas \( \theta_{DI} \) is given by the same equation that defines \( \theta_{DI} \) in the Balanced Equilibrium.

The continuity can be established by verifying that the equations for \( \theta_{DI} \) and \( \theta_{ID} \) for one equilibrium type coincide with another at each of the two cutoff values of \( A - e/\delta \).

**Proof of Proposition 4b** By (31), (34), the restrictions in the first column of Table 2, and (83), in the Selling Equilibrium,

\[ TV = \frac{e}{n^D} \left( n^D - n^{LD} + e \left( \frac{1}{\eta(\theta_{ID})} - 1 \right) \right). \]  

(114)

The result of the Proposition then follows, given that, by Proposition 4a, \( \theta_{ID} \) is decreasing in \( A \) in the Selling Equilibrium. By (34) and (32) and the restrictions in the second column of Table 2, in the Balanced Equilibrium,

\[ TV = \begin{cases} 
\frac{n^{LD} \eta(\theta_{ID})(1 - \mu(\theta_{DI}))}{n^{LD} \mu(\theta_{DI})(1 - \eta(\theta_{ID}))} & A \leq S + \frac{e}{\delta} \\
\frac{n^{LD} \mu(\theta_{DI})(1 - \eta(\theta_{ID}))}{n^{LD} \eta(\theta_{ID})(1 - \mu(\theta_{DI}))} & A > S + \frac{e}{\delta} \end{cases}. \]  

(115)

The result of the Proposition then follows given that, by Proposition 4a, both \( \theta_{ID} \) and \( \theta_{ID} \) are decreasing in \( A \) in the Selling Equilibrium. By (30), (32), the restrictions in the third column of Table 2, and (87), in the Buying Equilibrium,

\[ TV = \frac{e}{n^D} \left( n^D - n^{LD} + e \left( \frac{1}{\mu(\theta_{DI})} - 1 \right) \right). \]  

(116)

The result of the Proposition then follows given that, by Proposition 4a, \( \theta_{DI} \) is decreasing in \( A \) in the Buying Equilibrium.

Evaluate (114) and the first line of (115) at where \( A = B_M + e/\delta \) and (87) yields the same value of \( e \left( 1 - \frac{e}{n^D} \right) \). Evaluate the second line of (115) and (116) at where \( A = B_L + e/\delta \) and (88) yields the same value of \( e \left( 1 - \frac{e}{n^D} \right) \). This proves continuity.

**Proof of Proposition 4c** In the Selling Equilibrium, \( p \) is given by (74), which is increasing in \( \theta_{ID} \) and \( \theta_{DI} \). Given that in the Selling Equilibrium, \( \theta_{ID} \) is decreasing in \( A \) but \( \theta_{DI} \) is independent of \( A \), \( p \) must be decreasing in \( A \). That there is a discrete fall in \( p \) as the Selling Equilibrium turns into the Balanced Equilibrium can be established by showing that the denominator of (76), which gives \( p \) in the Balanced Equilibrium, is larger than that of (74) at any \( \theta_{ID} \) and \( \theta_{DI} \). Moreover, by (76), \( p \) in the Balanced Equilibrium is also increasing in \( \theta_{ID} \) and \( \theta_{DI} \), both of which are decreasing in \( A \). Finally, that there is a discrete fall in \( p \) as the Balanced Equilibrium turns into the Buying Equilibrium can be established by noting that the numerator of (76) always stays strictly positive.
Proof of Proposition 5a In the Balanced Equilibrium, \( \theta_{DI} \) is given by the solution to (97), whereas \( \theta_{ID} \) can be recovered from (96) once \( \theta_{DI} \) is known from the former equation. In the Buying Equilibrium, \( \theta_{DI} \) and \( \theta_{ID} \) are given by the solutions to (90) and (87), respectively. In the Selling Equilibrium, \( \theta_{DI} \) is given by the solution to (83), whereas \( \theta_{ID} \) can be recovered from (85) once \( \theta_{DI} \) is known from the former equation. The comparative steady states followed straightforwardly from these equations. Just as in the proof of Proposition 3a, the continuity can be established by verifying that the equations for \( \theta_{DI} \) and \( \theta_{ID} \) for one equilibrium type coincide with another at each of the two cutoff values of \( A - e/\delta \).

Proof of Proposition 5b In the Selling Equilibrium, \( TV \), given by (114) is decreasing in \( n^{LD} \), given that \( \theta_{ID} \) is independent of \( n^{LD} \) in the Selling Equilibrium. In the Buying equilibrium, \( TV \), given by (116) can be shown to be decreasing in \( n^{LD} \) with \( \theta_{DI} \) given in (90).

In the Balanced Equilibrium, \( TV \) is given by either the first or the second line of (115). To show that both expressions are increasing in \( n^{LD} \), we begin with noting that \( A^{D} \), by (79) and (80), in the first instance is given by

\[
A^{D} = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})},
\]

but by (97),

\[
A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} = n^{LD} + n^{D} - \frac{e}{\mu(\theta_{DI})}.
\]

Because \( \theta_{DI} \) increases with \( n^{LD} \) in the Balanced Equilibrium, the LHS strictly increases with \( n^{LD} \), and so dealers hold more inventory in total. The RHS, however, can only rise by less than the increase in \( n^{LD} \). Given that in the Balanced Equilibrium, \( A^{D} = n^{SD} + n^{LD} \), a larger \( n^{LD} \) must be accompanied by a smaller \( n^{SD} \). Also, according to (95), there would also have to be a smaller \( n^{SD} \). In the investor-dealer market, both dealer-sellers and dealer-buyers execute \( e \) trades in the steady state; i.e.,

\[
(n^{LD} + n^{SD}) = (n^{LD} + n^{SD}) \mu(\theta_{DI}) = e
\]

Because \( n^{SD} \eta(\theta_{ID}) \) and \( n^{SD} \mu(\theta_{DI}) \) strictly decrease with \( n^{LD} \), \( n^{LD} \eta(\theta_{ID}) \) and \( n^{LD} \mu(\theta_{DI}) \) must be strictly increasing in \( n^{LD} \). This implies that both the first and the second lines of (115) are strictly increasing in \( n^{LD} \).

The proof of continuity is as in Proposition 3c.

Proof of Proposition 5c In the Selling Equilibrium, \( p \) is given by (74), which does not directly depend on \( n^{LD} \), given \( \theta_{ID} \) and \( \theta_{DI} \). But then, the two market tightness in the Selling Equilibrium do not vary with \( n^{LD} \). The proof for the jump in \( p \) that occurs when the Balanced Equilibrium gives way to the Buying or the Selling Equilibrium follow from Proposition 4b.

Proof of Lemma A1 The equation for \( p_{Is} \) is from combining (15) and (68). The equations for \( p_{Ib} \) are from combining (17) and (73) for the Selling Equilibrium, (17), (68), and (75) for the Balanced Equilibrium, and (17), (78), and \( p = 0 \) for the Buying Equilibrium.
Proof of Propositions A1 and A2  By (47), given \( p, p_{ls} \) depends only on and is increasing in \( \theta_{DI} \). In the Selling and Balanced Equilibria, \( \partial \theta_{DI}/\partial A = 0 \) and \( \partial \theta_{DI}/\partial A < 0 \), respectively. Then, \( \partial p_{ls}/\partial A \) has the same negative sign as \( \partial p/\partial A \) in the two types of equilibrium. Next, in the Selling Equilibrium, \( \partial \theta_{DI}/\partial n^{LD} = 0 \), from which it follows that \( \partial p_{ls}/\partial n^{LD} \) has the same zero value as \( \partial p/\partial n^{LD} \). In the Buying Equilibrium, by Lemma A1, \( p_{ls} = p = 0 \). The discrete changes in \( p_{ls} \) at where one equilibrium type changes to another follows from the discrete changes in \( p \).

By combining (48) and (74), in the Selling Equilibrium,

\[
p_{IB} = \frac{\frac{1}{2} \beta (1 - \beta + \beta \eta (\theta_{ID})) (1 - \beta + \frac{\eta (\theta_{DI})}{2} \beta)}{1 - \beta + \frac{\eta (\theta_{DI})}{2} \beta} \left( 1 - \beta + \frac{\eta (\theta_{DI})}{2} \beta \right) (1 - \beta + \beta \delta - \delta \beta \eta (\theta_{DI}) \frac{\eta (\theta_{DI})}{2} \beta),
\]

where \( \partial p_{IB}/\partial \theta_{ID} > 0 \). Then, given \( \partial \theta_{ID}/\partial A < 0 \), it follows that \( \partial p_{IB}/\partial A < 0 \). Meanwhile, \( \partial p_{IB}/\partial n^{LD} = 0 \) holds given \( \partial \theta_{ID}/\partial n^{LD} = \partial \theta_{DI}/\partial n^{LD} = 0 \) in the Selling Equilibrium. That \( p_{IB} \) in the Buying Equilibrium, given by (50), does not vary with \( n^{LD} \) follows from \( \partial \theta_{ID}/\partial n^{LD} = 0 \) in said equilibrium. The discrete changes in \( p_{IB} \) at which one equilibrium type changes to another can be verified by checking how, given \( \theta_{ID} \) and \( \theta_{DI} \), \( p_{IB} \) in (48) exceeds \( p_{IB} \) in (49), which in turn exceeds \( p_{IB} \) in (50).

Proof of Lemma A2  The state variables of the planning problem (51) are \( \{ n_H^{ON} (t), n_L^{ON} (t), n^I_B (t) \} \), the initial conditions are \( \{ n_H^{ON} (0), n_L^{ON} (0), n^I_B (0) \} = \{ \tilde{n}_H^{ON}, \tilde{n}_L^{ON}, \tilde{n}_B^I \} \), the controls are \( \{ n_0^{SD} (t), n_1^{ID} (t), n_0^{LD} (t), n_1^{ID} (t), n_2^{LD} (t) \} \) and the equations of motion are, respectively,

\[
\begin{align*}
n_H^{ON} (t + 1) - n_H^{ON} (t) &= -\delta n_H^{ON} (t) + n^I_B (t) \mu (\theta_{ID} [t]), \\
n_L^{ON} (t + 1) - n_L^{ON} (t) &= \delta n_H^{ON} (t) - n_L^{ON} (t) \eta (\theta_{DI} [t]), \\
n^I_B (t + 1) - n^I_B (t) &= -n^I_B (t) - \mu (\theta_{ID} [t]).
\end{align*}
\]

The constraints are given in (22)-(26) that hold at each moment in time, which can be summarized by the following two equations:

\[
\begin{align*}
\theta_{ID} (t) &= \frac{n^I_B (t)}{n^D - n_0^{SD} (t) - n_0^{LD} (t)}, \\
\theta_{DI} (t) &= \frac{n^D + n^{LD} - A - n_0^{LD} (t) + n_L^{ON} (t) + n_H^{ON} (t)}{n_L^{ON} (t)}.
\end{align*}
\]

In the above, a pair of \( \{ n_0^{SD} (t), n_0^{LD} (t) \} \) uniquely determines the pair \( \{ \theta_{ID} (t), \theta_{DI} (t) \} \). This means that the controls can be stated in terms of the two market tightness only, whereby the admissible values are given by

\[
\begin{align*}
\theta_{ID} (t) &\in \left[ \frac{n^I_B (t)}{n^D - n_0^{SD} (t) - n_0^{LD} (t)}, \frac{n^I_B (t)}{n_S (n_H^{ON} (t), n_L^{ON} (t))} \right], \\
\theta_{DI} (t) &\in \left[ \frac{n_0^{LD} (t) + n_L^{ON} (t)}{n^D (n_H^{ON} (t), n_L^{ON} (t))}, \frac{n_0^{LD} (t) + n_L^{ON} (t)}{n_L^{ON} (t)} \right],
\end{align*}
\]

56
with $\pi^D_B$ and $n^D_B$ denoting, respectively, the largest and smallest possible measures of dealers-sellers and $\pi^D_S$ and $\pi^D_S$ denoting, respectively, the largest and smallest possible measures of dealer-buyers, given state variables $n^D_H(t)$ and $n^D_L(t)$. Note that:

1. To attain $\pi^D_H$, first allocate one unit each of the assets to be held by dealers $(A - n^D_H(t) - n^D_L(t))$ to either small or large dealers, and then allocate one more unit each to large dealers if $A - n^D_H(t) - n^D_L(t) > n^D$.

2. To attain $\pi^D_B$, first allocate two units each of the assets to be held by dealers to large dealers, and then allocate one unit each to small dealers if $A - n^D_H(t) - n^D_L(t) > 2n^D_B$.

3. To attain $\pi^D_B$, first allocate one unit each of the assets to be held by dealers to large dealers, and then allocate one unit each to either large or small dealers if $A - n^D_H(t) - n^D_L(t) > n^D_D$.

4. To attain $\pi^D_B$, first allocate one unit each of the assets to be held by dealers to small dealers, and then allocate two units each to large dealers if $A - n^D_H(t) - n^D_L(t) > n^S_D$.

To proceed, write (51) as

$$W(n^D_H(t), n^D_L(t), n^I_B(t)) = \max_{\theta_H(t), \theta_D(t)} \left\{ n^D_H(t) \nu_H + \beta W(n^D_H(t + 1), n^D_L(t + 1), n^I_B(t + 1)) \right\},$$

in which the state variables for $t + 1$ can be recovered from the equations of motions. There are four constraints corresponding to the four bounds of market tightness. Let $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ and $\lambda_4(t)$ be the respective Lagrange multipliers of the lower and upper bounds of $\theta_H(t)$ and the lower and upper bounds of $\theta_D(t)$.

Restricting attention to the steady state, we omit all time indices in the following. The first order conditions for $\theta_H(t)$ and $\theta_D(t)$ are then given by, respectively,

$$\beta n_B^H \mu'(\theta_H)(W_1 - W_3) + \lambda_1 - \lambda_2 = 0,$$

$$- \beta n_L^H \eta'(\theta_H)W_2 + \lambda_3 - \lambda_4 = 0. \tag{126}$$

In addition, there are three envelope conditions, one for each state variable:

$$W_1 = \nu_H + \beta(1 - \delta)W_1 + \beta \delta W_2 - \lambda_1 \frac{\partial(n_B^H / \pi^D_S)}{\partial n_H^D} + \lambda_2 \frac{\partial(n_B^H / \pi^D_L)}{\partial n_H^D}$$

$$- \lambda_3 \frac{\partial(n_B^H / n_L^D)}{\partial n_H^D} + \lambda_4 \frac{\partial(n_B^H / n_S^D)}{\partial n_H^D} \tag{128}$$

$$W_2 = \beta(1 - \eta(\theta_D))W_2 - \lambda_1 \frac{\partial(n_B^L / \pi^D_S)}{\partial n_L^D} + \lambda_2 \frac{\partial(n_B^L / \pi^D_L)}{\partial n_L^D}$$

$$- \lambda_3 \frac{\partial(n_B^L / n_L^D)}{\partial n_L^D} + \lambda_4 \frac{\partial(n_B^L / n_S^D)}{\partial n_L^D} \tag{129}$$

$$W_3 = \beta \mu(\theta_H)(W_1 - W_3) - \lambda_1 \frac{\lambda_1}{\pi_S} + \frac{\lambda_2}{\pi_S} \tag{130}$$

We first show that $\lambda_3$ must equal to 0. Suppose otherwise. By the definition of $\pi^D_S$, $n^D_S$, $\pi^D_B$ and $n^D_B$, $\lambda_1$, $\lambda_2$ and $\lambda_4$ must all equal to 0. Then equations (126) to (130) reduce to three
equations, (127), (128) and (129) with only two unknowns, $\lambda_3$ and $W_2$. The set of $\lambda_3$ and $W_2$ that satisfy all three equations is of measure zero. Therefore, $\lambda_3$ must equal to 0.

By the same argument, we can show that $\lambda_2$ must equal to 0.

Next, we prove that $\lambda_1 > 0$. Suppose otherwise. Together with the fact that $\lambda_2 = 0$, this implies $W_1 = W_2 = 0$. We already know $\lambda_3 = 0$. Then equations (126) to (130) reduce to three equations, (127), (128) and (129) with only two unknowns, $\lambda_4$ and $W_2$. We have reached the desired contradiction.

Finally, we prove that $\lambda_4 > 0$. Suppose otherwise. Then by equation (127), $W_2 = 0$. Plug it into equation (129), we must have $\lambda_1 = 0$, which contradicts our previous conclusion that $\lambda_1 > 0$.

To summarize, we have shown that $\theta_{ID} = \frac{n_L^{I}}{n_H^{I} - n_L^{I}}$ and $\theta_{DI} = \frac{n_H^{O}}{n_L^{O}}$. In other words, for efficiency, we should allocate the assets held by dealers to maximize the measure of dealers holding inventory and the measure of dealers having spare capacity: first allocate one unit each to large dealers; if $A - n_H^{ON} - n_H^{ON} > n^{LD}$, then allocate one unit each to small dealers; if $A - n_L^{ON} - n_L^{ON} > n^{D}$, then allocate one more unit each to large dealers.

**Proof of Proposition A3** The allocations as described in Lemma A2 are the same as the allocations as described in the discussions following Proposition 2.

**References**


58


