A Theory of Payments-Chain Crises

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Abstract

I introduce payment chains into a business cycle model. Consumption decisions are linked to each other in a chain of payments. Whereas some payments can be made immediately, other payments are postponed until other payments are executed. Delays in payments delay production. An unexpected contraction in some agents ability to obtain credit leads to a payments crisis. At an initial phase, a contraction in credit sends a mass of agents to a liquidity-constrained status. This delays their payments, but also slows down the payment of others, causing a cascade in delays in a chain of payments. The real effects of these delays is isomorphic to a drop in TFP, which in turn, shows up as a pecuniary externality. I characterize the transitional dynamics and characterize the sources of constrained inefficiency. For suitable preferences, the effects of an initial credit contraction may persist well after credit limits have been normalized.

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1. Introduction

Those of us who have lived through a credit crises can relate to the idea of a payments-chain collapse. Beyond the observable declines in credit conditions, there's also a general perception that transactions take longer: firms take more time to liquidate inventories, borrowers constantly postpone payments to creditors, and even workers may have to wait for their paychecks. All in all, there's a perception that the chain of payments has lengthened and that the speed of economic transactions has slowed down. In ways that are yet to be understood and better measured, these disruptions seem to carry pervasive effects on productivity. Most strikingly, the slow down in transactions seems to persist even several periods after the original credit tightening has eased. What at initial dates seems to be a driven by a credit crunch, eventually evolves into a payments-chain crisis.

This paper rationalizes the idea of payments-chain crises, within a business cycle model. The theory has three distinguishing features: First, the speed of economic transactions is a function of the length of the payments chain. Second, the length of the payments chain is a function of credit-market conditions and agent decisions regarding to carry out expenditures with spot transactions or transactions chained to their income. Third, a credit crunch slows down the speed of transaction, but their speed may remain low even after credit conditions are eased. These features are embedded into a business cycle model where I study how the consumption-savings decisions of agents endogenously determines the length of the payments chain in a give period. The theory showcases how payments chain disruptions manifest as a pecuniary externality and opens the door for some novel policy solutions.

Payment Chains. The staring point of the theory is a sub-model where expenditure decisions and production decisions are linked through a payments chain. This sub-model builds on a very natural observation: payments chains link the transactions of households. The expenditures of one household is the income of another household. In turn, the expenditures of a second household are the income of a third household, and so on. If some households cannot spend before they receive income and, moreover, have to wait between the time they receive income and spend it, they will transact with other households with some delay. Greater delays reduce the capacity to produce output efficiently, because production cannot start until some payments are made.

Of course, credit enables borrower households to spend before earning income. Hence, the provision of credit is a determinant of the speed of transactions. When credit conditions are so strong that any household can spend before earning income and there are no delays in production; when some when credit conditions are tight, transactions are postponed and output is lost.

To capture this idea, the starting point of the sub-model is a distribution of pre-determined transactions. Some of those transactions are programmed to be executed on the spot and others through chained payments. Unlike a Walrasian environment, but very much like in the spirit of money-search economy, transactions are not simultaneous (not centralized). On the contrary, nature randomly links different households through a payments-network of consumption and production (or expenditures and income): a household can only buy goods from another household. Transactions that are executed through the payments chain are transactions where income must proceed expenditures. By contrast, the spot transactions can be executed immediately, without the need to generate income. This sub-model produces a distribution of linked transactions where the length of payments chain is distributed geometrically with an endogenous probability that depends on the fraction of spot transactions. The greater the amount of spot transactions, the smaller the average payments chain and the faster the speed of production. I derive a formula that links output and TFP to the fraction of spot transactions.

The sub-model showcases that the distribution of means of payments determines the speed of transactions and average productivity. The presence of linked payments leads to externalities. The nature of the externality is that when an individual agent is able to transact earlier, because he makes spot payments, a second agent will earn income earlier. By earning income earlier, the second agent will transact at an earlier date, thereby speeding up the whole network of transactions.

This increases real income, and a greater and reinforcing demand for credit. A theme in the second part of the paper is that the distribution of spot and chained transactions is endogenous to household overall consumption and savings decisions. However, consumption and savings decisions do not internalize the effects of their on the payments chain, as these only show up as a pecuniary externality.

Payment Chains in a cycle. I embed the payments chain into a dynamic, yet tractable business-cycle model. In the business cycle model, some households are borrowers and some are lenders. To consume a desired amount, the household must program an amount of spot and chained transactions orders. If the household has too much debt, it will be constrained by the amount of spot transactions it can place; it will lack not have enough intra-period credit. Consumption obtained through chained transaction orders is more expensive, given the delays. The decision to borrow or lend is endogenous and depends on the real-interest rate and the distribution of spot and chained transactions of other households.

I demonstrate that in a steady state, households will not accumulate debt beyond the point where their spot transactions are constrained by their credit limits. Hence, the economy is efficient in a stationary equilibrium. However, this is not true during a credit crunch and the transition to steady-state that follows. A credit-crunch as a situation where agents suddenly lose access to credit lines that allow them to consume via spot transactions despite their debt holdings. During a credit crunch, borrowers must consume through chained transactions before they can spend. This delay, in turn, slows down the cash-flow of other agents and, thus, average productivity falls, leading to a *payments-chain crisis.*

A payments crisis has two phases: the initial credit crunch phase and an aftermath phase. During the initial phase, borrowers not only delay their transactions, but they roll-over their debts. When the credit-crunch is over, their debt remains high so they continue to consume with chained transactions. Thus, eventhough credit conditions have recovered, the payments-chain continues to be inefficiently delayed. The failure to internalize effects on the payments chain will explain why the speed of transactions is inefficiently slow, many periods after the initial financial disruptions were dissipated.

Policy Implications. The pecuniary externality induces three sources of inefficiency. First, savers consume too little relative, workers consume excessively with chained transactions, and finally, the real interest-rate is inefficiently high. These features are fleshed out when I study a Ramsey planner problem and articulate how a planner that respects the technology and takes interest-rates as given, would design its expenditure path during a crunch. The planner cuts back the consumption of borrowers, something that prompts a lower real interest than otherwise. This provokes a higher consumption path by savers, but since savers consume spot transactions, this reduces the overall length of the payments chains and improves outcomes. The planner trades-off consumption smoothing with the reduction in TFP.

In a final theme, I revisit some classic discussion on fiscal-monetary policy: I first study the benefits of a credit tax. I then highlight the beneficial effects of monetary transfers to borrowers, but show that these are more potent if they are used to spend rather than to repay debt. I finally study how government expenditures are beneficial if they are executed via spot transactions, but detrimental if the government does not pay instantaneously.

Organization. The paper proceeds as follows. Section 2introduces the mathematical formulation of payments chains. Section 3 embeds payments chains into an otherwise standard business cycle model with two agents. Section **??** presents an analytic characterization of the transitional dynamics in the model. Section 4 discusses the externality induced by payments chains. Finally, Section **??** connects the model with some studies on financial crises.

Literature Review

The paper falls in the cross-road of several branches areas of study in macroeconomics. First, it is connected to the literature that studies the relevance of payments (or monetary) frictions. Second, it connects to the literature on aggregate-demand externalities. Finally, it connects to the literature on endogenous network formation.

Payments Literature. A classic theme in monetary economics is how the quantity and distribution of real monetary balances affect economic outcomes. Lucas and Stokey (1987) analyzed a stochastic cash-in-advance economy in which changes in expected inflation distort real allocations as some goods must be purchased with non-interest bearing currency. Kiyotaki and Wright (1989) present a model with indivisible tokens which is explicit about their use as a means of payment. With some modeling assumptions, Lagos and Wright (2005) presents a model with arbitrary money holdings that is also explicit about the use of money as a means of payments. These papers are the spinal cord of the money-search literature that is interested in understanding the use of money as a payments instrument (see e.g. Lagos et al., 2011; Lagos and Rocheteau, 2009; Li et al., 2012; Nosal and Rocheteau, 2011; Rocheteau, 2011).[1] In common with this literature, in this paper I am interested in understanding how liquidity, in particular, its distribution, affects real allocations.

The most important departure in this paper is that mix between transactions programmed to carried out with credit and transactions expected to be executed after income is realized, endogenously create a payments chain. The length of this chain endogenously determines the average production time, which implicitly shows up as total factor productivity. Along those lines, Kiyotaki and Moore (1997) present an early model of a payment chain and study how monetary balances affect a sequence of transactions. A sequence of payments in a monetary model also appears in the three period model of Guerrieri and Lorenzoni (2009). Kalemli-Ozcan et al. (2015) extend a similar model to multiple rounds with discounting between periods. In our model, there are delays within the period that reduce output. These models have a chain that are predetermined in size. The novelty here is that the choice of payments, either with actual funds or with future funds, endogenously determines the size of payments chains and delays in production. Bigio and La'O (2020) also considers the propagation of financial shocks considering a general network of payments and production. That model is static and is about how credit spreads distort the allocation of inputs.

[1] Rocheteau et al. (2018) and Rocheteau and Rodriguez-Lopez (2013) are models where the use of money affect other spreads that, in turn, affect aggregate production. I

Another notable difference with classic monetary models is that most of these models, the distribution of financial wealth is degenerate. An exception is Rocheteau et al. (2016) which deals with a distribution monetary holdings, which in turn, creates a distribution of realized gains from trade. Green and Zhou (2002) studies the evolution of monetary balances in pair-wise meetings where trade results from a mechanism. Here, all transactions are pairwise, and the distribution of wealth determines the distribution of spot and chained payments over time, something that matters for TFP.

There is some earlier work studying the use of outside money to settle payments done with credit (see Freeman, 1996; Green, 1999). In this paper, all transactions are done with inside money and there's no need to settle transactions with outside money. Instead, transactions can be carried out, partially borrowing funds, or waiting for funds to arrive.

This paper also connects with the literature on networks. As in Oberfield (2018), the network is endogenous to economic decisions, but here, its endogenous to financial decisions. The paper also relates to models with network externalities, such as Elliott et al. (2014) or o Alvarez and Barlevy (2021). The novelty here is that the externality emerges as an inefficient network formation.

Some modeling features here can be traced to Shi (1997) and Lagos and Wright (2005) models. In common to Lagos and Wright (2005), periods are divided into a financial stage where savings and expenditures are decided, and then, a period where transactions and payments are executed. In common to Shi (1997), many decentralized transactions are carried out by the same households. Different from these models, transactions are linked through endogenous payments chains and there is no bargaining prior to transactions, but rather the transactions size is fixed.

Aggregate-Demand Externalities. In the model, there is a demand externality. An early model of demand externalities is Diamond (1982), where the consumption of some agents determines aggregate output. Most of the literature has focused on demand externalities that arise from nominal rigidities. Since the financial crisis of 2008, there's been interest in demand externalities arising in the context of financial crisis that as in Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2017). In these models, demand externalities emerge occur when a group of agents is suddenly cut back from the credit market, and cannot be corrected with monetary policy do to a zero-lower bound. Here, there are no nominal rigidities between periods; the nature of the externality is different. Moreover, the externality is two-sided: unconstrained agents don't internalize that by spending more, they help increase aggregate output as they speed-up "the payments-chain". Thus, as in the new-Keynesian model, there is insufficient demand by wealthy households. The other side of the externality is that debt-constraint agents delay production credit that delay transactions.

Another set of models consider pecuniary externalities, following Greenwald and Stiglitz (1986) and as studied in detail by Davila and Korinek (2018). A feature of the model here is that the externalities created in the formation of payments chains indeed show up as pecuniary externalities. This opens the door for the desire to introduce corrective taxes when wealth cannot be distributed immediately. A novelty of the paper is that when I consider fiscal policy, the way in which expenditures are financed, matters. Korinek and Simsek (2016) study optimal ex-ante interventions in models with ex-ante aggregate demand externalities. There is likely room for such policies in this environment too, but this is something I leave with future research.

The closest papers to this one are La'O (2015), Woodford (2022), and Guerrieri et al. (2020). With respect to La'O (2015), transactions are structured in a circular flow of payments that lasts forever. The allocation of funds in the ring determine production. Here, the distribution of funds determines a distribution of payments whose length determines output. In turn, Woodford (2022) and Guerrieri et al. (2020) consider the inefficiencies in economies with payments in the context of price rigidities. In the present model, the prices are rigid only within the period. Hence, there's no scope for an interest rate policy and the zero-lower bound plays no role. However, as stressed in these papers, there is scope for fiscal policy or transfers. Different from these models, the network of transactions here is endogenous and a function of consumption-savings decisions.

2. Payment-Chains and Productivity

This section presents a simple environment meant to capture the main idea: delays in the payments chain and its connection to production. I then adapt the environment of this section and introduce payment chains into the dynamic business cycle model that appears below.

Environment. I study a collection payments which generate income (production) and expenditure (consumption) relations. These payments are linked to other payments through a payments chain. Time τ runs continuously over a unit interval $[0,1]$. All transactions in this chain occur within that time. The economy is populated by an equal number, N, of workers and shoppers. Workers and shoppers are assigned an identity $i \in \mathcal{N} = \{1, 2, \ldots, N\}$. I describe the environment for a finite N to fix ideas, but derive results for $N \to \infty$.

I will define shopper-worker relations that, when put together into a payment chain, will shape production. The production relation $\mathcal{P}$ is a one-to-one assignment between shoppers and workers. In this relation the worker's output can only be consumed by its shopper and the shopper can only buy goods from its worker. For reasons that will become clear later, a pair of shopper and worker related through $\mathcal{P}$ cannot have the same identity.[2]

The income relation is the union of two other relations. One is the couple relation $\mathcal{X}$ where the shopper and worker in these couples have the same identity. The shopper in this couple does not have funds at $\tau = 0$ and once the worker gets paid, at some $\tau > 0$, she can transfer the funds to the shopper.

In contrast to the production relation, in the income relation there are workers and shoppers, not coupled, called singles.[3] A single shopper shows up to the payments chain with a means of payment at $\tau = 0$. This implies that they can place a shopping order immediately at time $\tau = 0$. Thus the singles relation $\mathcal{S}$ is

[2]Formally this is a bijection $\mathcal{P} : \mathcal{N} \to \mathcal{N}$ such that if $\mathcal{P}(i) = j$ then shopper i buys from worker j and $i \neq j$.

[3]Formally, let $A \subset \mathcal{N}$ be the set of couples, the couple relation is the identity bijection on A, $\mathcal{X} : A \to A$

the identity relation over the set of singles.[4] The income relation then is the union of both relations $\mathcal{I} = \mathcal{X} \cup \mathcal{S}$ and is simply the identity relation.

For now, we take their relative populations as given. We assume that a fraction μ of the N same identity worker-shopper pairs is coupled. In the dynamic model below, this fraction is endogenous and is derived from household decisions.

Payment Chains. Notice that, whereas identity defines the couple relation, we let nature choose among the possible production relations $\mathcal{P}$ with equal probability. These two underlying relations produce a network of payment chains, which is my ultimate object of study. Consider a single shopper i related with worker j (i.e. $\mathcal{P}(i) = j$), this creates a payments link because shopper i will pay worker j for producing goods. If worker j is single, any funds it receives will remain with her and there are no more payments link. Thus, this chain of payments would have length zero. However, if j is coupled she will transfer the funds to its shopper. This creates a link between shopper i and shopper j. In turn, shopper j is related to worker k (i.e. $\mathcal{P}(j) = k$), thus, the funds received from shopper i that were transferred to shopper j will be transferred to worker k. If k happens to be single, then there are no more chained payments. As a result, this chain of payments would have had length one. However, if k happens to be coupled, she will transfer funds to her shopper, and the chain will continue. Notice that the length of the payments chain is the number of links between consecutive shoppers. Below we will see that this is equal to the number of couples in a payment chain.

Characterization. A payment chains network is a sequence of payment relations where a shopper buys from a worker and the worker, possibly, funds its couple's consumption which will generate a payment to a second worker and so on and so forth until a coupled shopper pays to a single worker and the payment chain will end and restart from the single shopper.

Armed with the relations, we formally define a payment chains network by induction.

[4]The singles relation is the identity bijection over A^c, $\mathcal{S} : A^c \to A^c$.

Definition 1. *A **payment chains network** is a sequence of identity pairs such that the successor of any (i,i) is the identity pair (j,j) corresponding to the worker that sells to shopper i (i.e. $\mathcal{P}(i) = j$). Further, for any pair (i,i) we can say whether it is coupled or single.*

Intuitively, the network advances because shoppers pay to workers that are immediately to their right and the payment chain continues if the worker is coupled.

Since the pairs are identity pairs we could characterize the network through a **sequence representation** by a sequence of identities $\mathcal{R} = \{i_n\}_{n\geq 0}$ where $i_{n+1} = \mathcal{P}(i_n)$. For example, $\mathcal{R} = \{\ldots, i, j, k, l, m, n, o, p, \ldots\}$.Namely, j is the worker of shopper i in the production relation, k is the worker of shopper j and so on. We can further characterize the network $\mathcal{R}$ as through a **binary representation** by a sequence of singles and couples $\mathcal{B} = \{b_n\}_{n\geq 0}$ where $b_n = s$ if the n-th pair is single and $b_n = x$ if the n-th pair is a couple. For example, $\mathcal{B} = \{\ldots, s, s, x, s, x, x, x, s, x, \ldots\}$ where i and j are singles, k is coupled and so on. Notice that whenever a single succeeds a couple $(\ldots, x, s, \ldots)$, that meeting in the network represents the end of one payment chain because the couple transfers funds to a single worker who will not further transfer funds. When a single succeeds a single $(\ldots, s, s, \ldots)$, it means that a chain of payments has length zero, because no two shoppers are linked. In turn if a couple succeeds a single $(\ldots, s, x, \ldots)$ there is a payment chain of length one because the single shopper will pay the coupled worker whom, in turn, will transfer funds to its shopper creating a link between the single and coupled shopper. An induction argument shows that the length of a payment chain $\{\ldots, s, x, \ldots, x, \ldots\}$ with n consecutive couples is n.

We can further represent a network by rewriting its binary representation $\mathcal{B}$ as a **collection of payment chains**. Under this representation it only makes sense to consider networks that start with a single.[5] A **payment chain** is a finite[6] subsequence of the sequence representation of the network that starts with a single and finishes before the next single. The network in our example reduces to an ordered

[5]Because if one starts with a couple the first shopper has no means of payment and its worker couple will not produce (and will not get paid) because nobody is buying from her.

[6]I assume that a next single will always appear.

collection of payment chains $\mathcal{C} = \{\ldots, \mathcal{C}_n, \mathcal{C}_{n+1}, \mathcal{C}_{n+2}\ldots\}$ where $\mathcal{C}_n = \{s\}$, $\mathcal{C}_{n+1} = \{s, x\}$, $\mathcal{C}_{n+2} = \{s, x, x, x\}$.

Examples. I present the following figures to illustrate the concept of a payments-chain. In this case we have $N = 8$, the production relation is characterized by the sequence $\mathcal{R} = \{1, 5, 7, 4, 6, 2, 3, 8\}$ and the income relation is characterized by $\mathcal{B} = \{s, x, s, x, x, x, s, x\}$ as explained above (e.g. shopper 1 buys from worker 5 and 1 is a single and 5 is a couple). In figure 1 we can see the income relation, in green we have the shopper-worker pairs that are coupled and the arrow from worker to shopper depicts the flow of income in this direction. In contrast, we have in blue, pairs that are single and do not exhibit a flow of income between worker and shopper.

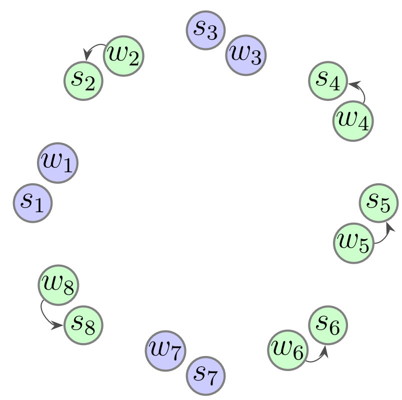

Figure 1: Income relation

In figure 2 I add the production relation which creates (jointly with the income relation) payment-chains. To interpret the graph, notice that the chain starts with the (single) shopper 1 who pays for production of worker 5 which is coupled and will transfer funds to shopper 5 who in turn pays for production of worker 7. However, worker 7 is single so the (blue) payment chain will end at this point. This first payment chain has length one because it created one link between two shoppers (1 and 5), the length of the chain is also equal to the number of couples in the chain. With shopper 7 a new (orange) payment chain will start. This chain links shoppers 7, 4, 6 and 2 because 4, 6 and 2 are couples. Naturally, the length of this chain is 3. Now, the last shopper, s_2, of the second (orange) payment chain pays for production of worker 3 who stops the chain since it is single. This starts the last chain which links shopper 3 and 8.

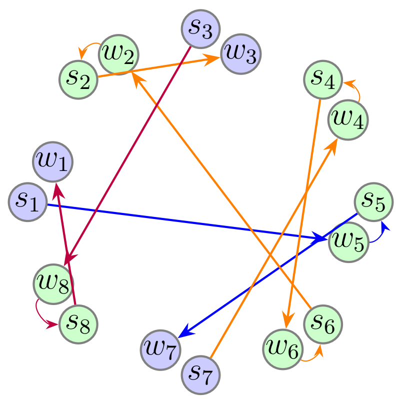

Figure 2: Payments-chain network

I emphasize that in this finite case I chose to "close" the payment chain by requiring that shopper 8 buys from worker 1, this implies that the same chains will happen again and again. I will rule out this type of "cyclying" behavior in the infinite case. Namely, I will require that the payment chain advances (to new shopper-workers) and does not create a loop among the same shopper-workers.[7]

Figure 3 summarizes the information in figure 2 using the fact that the income relation is the identity. Again, the first (blue) chain starts with 1 and will continue up to 7 since 5 is coupled. The second chain (orange) starts at 7 and will continue up to 3 since 4, 6, 2 are coupled. The last chain starts with 3 and continuous up to 1 because 8 is coupled. This figure emphasizes the key object of study: length of chains. Abstracting from the underlying structure of the network we clearly see the number of links between shoppers for each chain.

Finally, figure 4 shows the shoppers in the left and workers in the right, the

[7]This is necessary to have randomness in chain length. Formally, this requires that nature chooses the worker i_{n+1} for shopper r_{n} among $\mathbb{N}-\left\{r_{1},\ldots,r_{n}\right\}$. Since this is still an infinite set, this drawing remains independent.

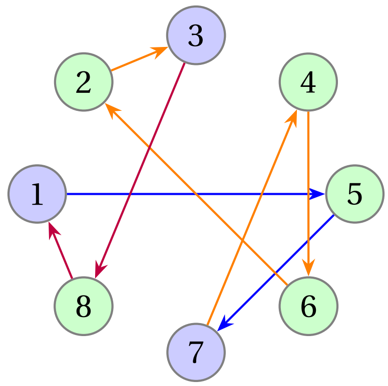

Figure 3: Summary of a payments-chain network

production-related payments are depicted by arrows directed to the right and the income-related payments are depicted by arrows directed to the left. For single pairs there is no flow of income (to the left) and this gives rise to new chains. In this case it is also very easy and intuitive to pick up the chain length: it is the number of colored arrow tips (for each chain) since they reflect the number of linked shoppers within a chain. We easily see that the first chain has length 1, the second length 3 and the third one length 1.

Let's now derive the distribution of payment chain lengths for $N \to \infty$. Recall that nature assigns production relations randomly. Thus, the distribution of length n of the payments chains follows some probability mass function depending on the proportion of couples μ as a parameter, namely some $G(n;\mu)$. In particular, allowing $N \to \infty$ and standing in any node of the network the probability of the next identity to be coupled or single is μ and $1-\mu$ respectively. Namely, the next type distribution is independent and identically distributed as a Bernoulli trial with probability μ. Now, a chain of payment is of length zero if the starting single (which is given) is followed by a single, this happens with probability $1-\mu$. Likewise, the chain is of length 1 if the first draw after the first single is a couple, which happens with probability μ, and the second draw is a single, which happens with probability $1-\mu$. The chain is of length two if there are two consecutive draws of couples followed by a single, and this occurs with probability $\mu^2(1-\mu)$.

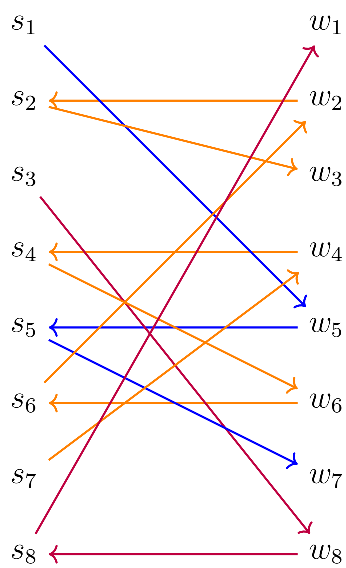

Figure 4: Payments-chain network

Proceeding by induction, we arrive at the following result:

Proposition 1. Let $n \in \{0, 1, 2, \ldots\}$ be the length of a payment chain, then n distributes geometrically with parameter μ, i.e., $G(n; \mu) = (1 - \mu)\mu^n$ is the probability mass function of n.

We use this distribution to solve for TFP and output once we consider how payments induce delays in the production chain. In the model, what really matters is the distribution of lengths of the payment chains starting from from a single shopper up to the first single worker.

Orders, Payments, Transfers, and Production Protocols. So far we have remained silent about how and how much do shoppers pay workers and what and how much is produced. Next, I answer these questions. Shoppers uses tokens worth one unit of labor as a means of payment. As in Kiyotaki and Wright (1989), tokens are indivisible. Thus, tokens can only be used to purchase one unit of labor. At the time the shopper worker relationships are realized, shoppers agree to transfer the funds to the worker in exchange for his production, whatever its production is. All workers will carry out some amount of production. Thus, notice that there's one unit of labor employed for every shopper and since the production relation is a one-to-one map, the labor market will clear.

What is special about the environment is that the time where production takes place matters for the total production generated each period. Hence, the times at which transactions takes place will impact TFP. There are two key assumption. First, production cannot begin unless the shopper transfers tokens to the worker. Second, recall that coupled workers need to transfer funds to their coupled shopper partners. The second assumption is that the worker can transfer the tokens only after the fraction $1-\delta$ of his production is done.

Let's now discuss how production takes place. The worker has one unit of labor endowment per instant of time in the interval $[0,1]$. Since production cannot begin until she is paid, she starts production at some endogenous random time τ. Thus, she has $\sigma \equiv 1-\tau$ time available to begin produce. Then, her output will be given by: $Y_\sigma = \sigma$.

Let's see what this means for the time at which production can begin to take place for different workers in different chains. If the chain is of length 0, production can begin immediately, so the time to build is 1, so her production is 1. Next, consider a chain of length 1 $\{s, x\}$; the shopper pays the coupled worker and starts production immediately but she only transfer the token funds to its couple shopper at time $1-\delta$ when the time to build is δ. Next consider a chain, $\{s, x, x\}$ in which the second worker starts with δ time left to build so it produces δ and accomplishes $1-\delta$ of its production at a time that leaves δ^2 time to build for the following worker.

We see a pattern for the finishing time of transfer-required production for a chain of length n. The required production seems to end at time $1-\delta^n$ leaving δ^n time for the following worker. We see that this holds for $n=1$ and if the n-length chain finishes required production at time $1-\delta^n$ the next worker will finish its required production at time $1-\delta^n+(1-\delta)\,\delta^n = 1-\delta^{n+1}$ leaving δ^{n+1} time for the following worker. This proves that the time to build in a chain of length n are $\{1, \delta, \delta^2, \ldots, \delta^n\}$ where the k-th element in this sequence represents the time to build available to the $k-th$ worker in the sequence.[8] We can generalize production along many directions, but it is convenient to keep things simple this way to convey the main idea.

Statistical Properties. Our next goal is to compute the expected output for single and coupled workers given μ and the distribution of lengths I derived in Proposition 1. In turn, this allows us to compute total output as a function of μ and δ. The following Proposition provides the results.

Proposition 2. (Output per worker): *Given μ and $N \to \infty$, the expected output of a worker in a production relation with single shoppers is 1. The expected output of a worker in a production relation with coupled shoppers is*

$$\mathcal{A}(\mu;\delta) = \frac{(1-\mu)}{\mu}\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right) < 1. \tag{1}$$

Furthermore, $\mathcal{A}_{\mu}(\mu;\delta) < 0$ and satisfies the following limits:

$$\lim_{\mu\to 0}\mathcal{A}(\mu;\delta) = \delta \textit{ and } \lim_{\mu\to 1}\mathcal{A}(\mu;\delta) = 0 \textit{ and } \lim_{\delta\to 0}\mathcal{A}(\mu;\delta) = 0 \textit{ and } \lim_{\delta\to 1}\mathcal{A}(\mu;\delta) = 1.$$

In turn, average output per hour is:

$$\mathcal{Y}(\mu) = (1-\mu) + \mu\mathcal{A}(\mu) < 1.$$

Furthermore, $\mathcal{Y}_{\mu}(\mu) < 0$ and satisfies the following limits:

$$\lim_{\mu\to 0}\mathcal{Y}(\mu;\delta) = 1 \textit{ and } \lim_{\mu\to 1}\mathcal{Y}(\mu;\delta) = 0 \textit{ and } \lim_{\delta\to 0}\mathcal{Y}(\mu;\delta) = 0 \textit{ and } \lim_{\delta\to 1}\mathcal{Y}(\mu;\delta) = 1.$$

Finally, $\mathcal{A}$ and $\mathcal{Y}$ are concave in μ.

This theorem is the key theorem in the paper. Total factor productivity, $\mathcal{Y}(\mu;\delta)$ that is, total output per worker, is a decreasing and concave function of the proportion of expenditures that are not paid upfront. To construct, $\mathcal{Y}(\mu;\delta)$ I first I

[8] Though bare in mind that in a n-length chain the n-th worker will be a single worker and it does not appear in the binary representation since it always ends with the last couple.

compute the per-worker production in a n-length chain, $(n^{-1}\sum_{i=1}^{n}\delta^{i})$. Then, I compute the expected value using the probability mass function conditional on being in a chain of non zero length $((1-\mu)\,\mu^{n}/\mu)$. Then, I integrate across all possible lengths to obtain the formula for $\mathcal{A}(\mu;\delta)$. Under this formula, then, $\mathcal{Y}(\mu;\delta)$ is constructed by noting that the fraction of chains of length zero is $(1-\mu)$ and the have production 1, where in the chains with at least one coupled payment, average output is $\mathcal{A}(\mu)$.

With respect to the limits of $\mathcal{A}$, as $\mu \to 1$ the chains of couples become larger and larger but the per-worker production decreases to zero, since the additional production of coupled workers decreases exponentially. On the other hand, when $\mu \to 0$ the chain length tends[9] to 1 and thus its per-worker productivity tends to δ because production will be delayed by at most one period. In turn, when $\delta \to 0$ the required amount of production to transfer funds tends to 1 and the worker related to the coupled shopper has no time left to build so the per-worker production is zero. Conversely if $\delta \to 1$, one could think coupled shoppers as singles since they can obtain funds immediately, naturally per-worker production tends to one since time left to build does so for each worker.

It is worth discussing the sources of TFP losses in this economy. Is there a congestion externality? No. Every transaction generates a production order. Is it the fact that transaction amounts are fixed? In the model prices are fixed per transaction, and objects production contracts fixed quantities. Even if payments would be negotiated, the quantity transacted would be the same. Thus, the fact that the transactions price is predetermined, is irrelevant for production amounts. What's causing the loss in output is the delay in payments and the misallocation of liquidity across the chain. If every transaction would take place instantaneously, then, no output would be lost. But in addition to that, at the margin, and additional unit of chained expenditures would at least lengthen one chain and this reduces production time. Even if the number of spot payments is fixed, a planner would like to distribute singles so that each chain is of the same length, simply because δ^{n} is a convex loss.

[9]Since we conditioned to at least length 1 chains

Finally, define $q(\mu;\delta) \equiv \mathcal{A}(\mu;\delta)^{-1}$, as the shadow price to purchase a unit of the good using chained consumption. Naturally, q is increasing and convex in μ and represents the cost of acquiring goods via chained transactions. As an illustration, 5 plots the production as a function of the chained expenditures ratio. Note how productivity quickly plummets as chained consumption increases. The lower δ, the lower the productivity.

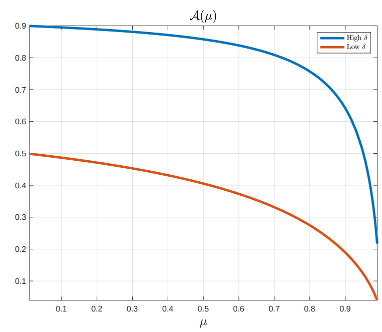

Figure 5: TFP as a function of chained expenditure ratio.

Note: This figure plots $\mathcal{A}$ corresponding to $\delta = 0.9$ and $\delta = 0.5$.

Discussion: Reduction of Economic Complexity. The model of payments chains is as simple as possible. In practice, economies involve a large variety of transactions that differ in size. Incorporating this richness along that transactions heterogeneity dimension and how that translates into other payments would lead to highly complex combinatorial problem. However, statistical physics handles similar levels of complexity when studying the dynamics of interacting particles.[10]

In an economic environment, it is natural for agents to negotiate the price of transactions or turn down transactions of delays are too long. Studying such interactions would make the problem more interesting but complicate the analysis.

Despite being simple, the problem here illustrates the effects of delays in payment chains. A virtue of the approach is that we have obtained a mapping from a financial quantity, μ, to TFP. Because this mapping is derived, and depends on

[10]In particular, Feynman diagrams are a tool to calculate statistical properties of quantum phenemonen.

agent decisions, as we will see next, this allows us to study policy exercises in a way that is immune to the Lucas critique.

Discussion: Financial Micro-Foundations. The model relies on three important assumptions: First, production has to wait until payments are made. Second, income can be transferred to perform payments with some delays. Finally, shoppers do not back-out from a production meeting even though the transaction size is fixed, and the amount of production depends on the time to build. All three assumptions, can be rationalized with deeper micro-foundations.

To rationalize why production must wait until some means of payment is shown, we can consider transactions in which the goods are customized to the shopper. For example, this would be the case construction contract. To rationalize the delay in the transfer of funds, continuing with construction example, the client may want to see part of the work done before transferring the funds. Finally, we can argue that the transaction is fixed because the worker must be compensated for his outside option. Otherwise, he could have found another match. Breaking the contract could have a cost in terms of reputation.

3. Payment-Chains in a Business-Cycle Model

We now incorporate payment chains into business cycles. The section lays out the environment taking the results from the previous section as given.

3.1 Environment

Timing. The timing has some special features. Time is continuous and the horizon infinity. Expenditure and savings decisions take place at integer dates, $t = \{0, 1, 2, ...\}$. Transactions and production take place between the integer dates. The interval between each integer corresponds to the time interval studied in the previous section. The economy features is of perfect foresight, but I will study an unanticipated shock. Labor units play the role of a numeraire.

Demographics. The economy is populated by two classes of big-family households. One class are working-class households (workers) that are rich in human wealth, but have negative financial wealth. The other class are financially wealthy households (savers). Both households are identical except that savers do not supply labor. Both households discount utility over time at rate β. As in, (Lucas,), each household class is composed of a continuum of members that take care of an expenditure an income.

Income, Expenditures, Transactions, and Prices. Next, I describe how expenditures and income are generated. In turn, total production and consumption depend on expenditures. The worker household has a large number of workers $\mathcal{N}$ with mass 1—as in the previous section, each worker supplies $1/\mathcal{N}$ labor units so that the total labor supply is one. The labor supply is inelastic. Both households choose a quantity S of goods that are purchased with spot transactions. For that, households must bring liquid balances. The also chose a quantity X of goods consumed by executing chained transactions. For that, households chain the expenditures in these goods, with one unit of labor income.

To obtain the corresponding quantities of spot and chained goods at t, households must spend E_t^x in transactions that lead to chained consumption and E_t^s on spot transactions. Let E_t be the total expenditures at t, $E = E^x + E^s$. As in the previous section, each transaction is of fixed size and the payment is in one unite of labor. Then, since total units of labor are employed, labor income must equal 1. Since all transaction are quoted in one unit of labor, the, $E_t = 1$. Since each transaction involves one unit of labor, then, the number of transactions is also $\mathcal{N}$.

Letting $\mathcal{N} \to \infty$, then,

$$\mu_t = E_t^x / \left(E_t^x + E_t^s\right) = E_t^x.$$

Then, given μ, total output is obtained as in the previous section:

$$\mathcal{Y}\left(\mu_t\right) = S_t + X_t.$$

Since, we know that all the transactions that are spot produce one unit of output, we have that $E^s = 1$. Moreover, the fraction of spot transactions is $1 - \mu$. Thus, we have that chained consumption must satisfy, $X = \mathcal{A}(\mu)\,\mu$. Since the total fraction of chained payments is μ_t, we have that:

$$q(\mu_t)X_t = \mu_t \text{ where } q(\mu) \equiv \mathcal{A}^{-}(\mu).$$

From the perspective of the household, its total expenditures then defines a "shadow" price $q(\mu)$ that it takes as given:

$$S + q(\mu)X = E.$$

Saver Household. Households consume and save in deposits that earn a deterministic real return R_t. The period utility is $\log(\cdot)$ over a sequence of consumption $\{C_t\}$. Saver households start each date with an amount of real deposits, D_t as their only source of wealth. Then, they choose between an amount of savings for the future, D_{t+1}, and their current expenditures in consumption goods. Since households consume strictly less than their wealth and they don't earn any labor income, their expenditures are all spot consumption. The saver's problem is given by:

Problem 1. (Saver's Problem): *Given D_0 and the path or real interest rates $\{R_{t+1}\}_{t\geq 0}$, wealthy households chose a sequence of savings $\{D_{t+1}\}_{t\geq 0}$ to maximize,*

$$\max_{\{D_{t+1}\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log(C_t^s),$$

subject to the following budget constraint:

$$\frac{D_{t+1}}{R_{t+1}} + C_t^s = D_t, \ \forall t \geq 0.$$

The solution to this problem gives us the savers consumption path, C_t^s. Naturally, these consumption equals expenditures.

Since wealthy household's must maintain positive wealth D_t to sustain positive consumption, they will always maintain positive savings. Working household's must therefore be the borrowers in this economy. Since the economy is closed, clearing on the financial markets requires:

$$D_t = B_t, \tag{2}$$

where B_t are the borrowing of workers. Thus, I proceed to the worker's problem treating B_t as worker debt.

Worker Household. Workers start each date with an amount of debt, B_t. Workers supply a total of one of units of labor inelastically. The period utility of workers is also $\log(\cdot)$. Consumption C_t^w is the sum of goods obtained through spot expenditures, S_t^w, and chained expenditures, X_t^w:

$$C_t^w = S_t^w + X_t^w. \tag{3}$$

This distinction is important because whereas S_t^w has a unit price, consistent with the production cost. In turn, chained consumption X_t^w has an implicit price q_t that, in turn, depends on the total distribution of spot and chained payments.

Borrowing B_t is limited by the natural debt limit, $\bar{B} = 1/\beta$, in order to prevent Ponzi schemes. The debt limit will never bind because it would mean infinite negative utility to the worker. To execute spot transactions, S_t^w, the worker must borrow within the period it wants to execute spot transactions at all. Intra-period debt carries no interest, as the debt is repaid within the period. The repayment is always feasible because for each unit of spot expenditures there is at least on unit of labor income. However, intra-period debt is limited. Namely, in addition to natural debt limit that applies to *inter-period* debt, I introduce a different constraint, a *borrowing limit,* $\tilde{B}$—we refer to the limit as the SBL, for spot borrowing limit from now on. The SBL $\tilde{B}$ caps the amount of *intra-period* borrowing. Namely, spot transactions are capped as follows:

$$S_t^w \leq \max\left\{\tilde{B} - B_t, 0\right\}. \tag{4}$$

Of course, the worker can execute chained transactions in which case he does not have to borrow intra-period. However, chained consumption is costly because $q_t \geq 1$.

Note that between period, the volume of credit is B_t but within period, $\tilde{B} - B_t$ is borrowed. We turn not turn to the worker's problem.

Problem 2. (Workers's Problem): *Given B_0 and the path or real interest rates and SBL's, $\left\{R_{t+1}, \tilde{B}_t\right\}_{t\geq 0}$, the worker chooses a sequence spot expenditures S_t^w and chained expenditures X_t^w to maximize:*

$$\max_{\{s_t, x_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log\left(C_t\right),$$

subject to the flow budget constraint:

$$B_t + S_t^w + q_t X_t^w = \frac{B_{t+1}}{R_{t+1}} + 1,\ \forall t \geq 0, \tag{5}$$

the definition of total consumption (3), the constraint on spot transactions (4), and the natural debt limit, $B_t \leq \bar{B}$.

Equilibrium. Given the expenditure choices of both households, $\{X_t^w, S_t^w, C_t^s\}$, the ratio of chained expenditures at t is:

$$\mu_t = q\left(\mu_t\right) \cdot X_t^w.$$

This ratio defines total production, TFP, and the effective cost of consumption obtained with chained expenditures at time t:

$$Y_t = \mathcal{Y}\left(\mu_t\right),\ A_t = \mathcal{A}\left(\mu_t\right),\ \text{and}\ q_t = \mathcal{A}\left(\mu_t\right)^{-1}. \tag{6}$$

In the economy, the goods market clearing condition is thus:

$$C_t^s + S_t^w + X_t^w = \mathcal{Y}\left(\mu_t\right), \tag{7}$$

and furthermore, from here we derive the following income-expenditure identity:

$$C_t^s + S_t^w + q_t X_t^w = 1. \tag{8}$$

I define an equilibrium as follows.

Definition 2. *An equilibrium is a sequence of asset positions and expenditures* $\{B_t, D_t, C_t^s, S_t^w, X_t^w\}$ *together with a sequence of real rates and implicit prices* $\{R_t, q_t\}_{t\geq 0}$ *such that:*

1. *Given* $\{R_t, q_t\}_{t\geq 0}$, $\{B_t, S_t^w, X_t^w\}$ *solves the worker's problem and* $\{D_t, C_t^s\}$ *solves the saver households problem.*

2. *The asset market and goods market clears: (2) and (7).*

3. *The price* q_t *is consistent with the ratio of chained-transactions (6).*

I now proceed to the characterization of equilibria.

3.2 Characterization

Solution to household problems. The wealthy household's problem is standard and its solution yields the standard constant marginal propensity to consume, characteristic of log utility. I summarize it in the following proposition:

Proposition 3. *Let* D_0 *be given. The solution to the wealthy household's problem,* $\{D_{t+1}\}_{t\geq 0}$, *is given by:*

$$\frac{D_{t+1}}{R_{t+1}} = \beta D_t, \ \ \textit{and} \ C_t^s = (1-\beta)\, D_t \ \ \forall t \geq 0. \tag{9}$$

We know that the worker can never consume above his labor income—combine $C^s > 0$ with the goods clearing condition. The worker's problem is more complicated because consumption depends on the fraction of spot transactions μ. However, in essence, once the worker decides its savings and expenditure, and for a given level of expenditure, it maximizes the spot consumption because its cheaper. Thus, we can split a desired level of consumption, which satisfies $C_t^w = S_t^w + X_t^w$, into following spot and chained consumptions:

$$S_t = \min\left\{\max\left\{\tilde{B} - B_t, 0\right\}, C_t^w\right\} \tag{10}$$

and

$$X_t = C_t^w - \min\left\{\max\left\{\tilde{B} - B_t, 0\right\}, C_t^w\right\}. \tag{11}$$

Naturally, $B_{t+1} = R_{t+1}\left(B_t + S_t^w + q_t X_t^w - 1\right)$. After we have solved the optimal expenditure composition, given total expenditures, we obtain a generalized Euler equation.

Proposition 4. (Workers's First-Order Condition): *Consider a sequence* $\{R_t, q_t\}$. *Any solution to the worker's problem,* $\{B_{t+1}\}_{t\geq 0}$ *such that* $\tilde{B}_t - B_t \neq C_t^w$ *and* $\tilde{B}_t = B_t$ *satisfies the following generalized Euler condition:*

$$\frac{C_{t+1}^w}{C_t^w} = \beta \frac{R_{t+1}}{\tilde{q}_{t+1}^s / \tilde{q}_t^e} \tag{12}$$

where

$$\tilde{q}_t^e \equiv q_t \mathbb{I}_{[X_t>0]} + \left(1 - \mathbb{I}_{[X_t>0]}\right) \text{ and } \tilde{q}_{t+1}^s \equiv q_{t+1}\mathbb{I}_{[S_{t+1}=0]} + \left(1 - \mathbb{I}_{[S_{t+1}=0]}\right). \tag{13}$$

The proof is not immediate from the Envelope Theorem because the budget constraint features a kink. Yet, the solution has a clear interpretation.[11] The left-hand side is the usual marginal rates of substitution between t and $t+1$ consumption. The right-hand side captures the usual relation between discounting and rate of return, βR_{t+1}, but there is an additional term, the marginal inflation. The marginal inflation captures the ratio of the relevant prices that enter marginal decisions: the marginal price of expenditures at t, $\tilde{q}_t^e$, and the marginal price of goods consumed after savings brought to $t+1$, $\tilde{q}_{t+1}^s$.

At an optimal solution, if chained goods are consumed at t, any saving will cut back the expenditures of chained goods. Since the price of chained goods at

[11] To proof this result, I use a relaxation method: I assume c_t^w is chosen with some noise, ε, and take the noise to zero—one can also use directly a Lagrangian method. Let's provide the intuition behind the Euler equation.

t is q_t, the cost of delivering a unit of current marginal utility will be q_t if at least some chained goods are consumed. Otherwise, the marginal price is 1. Hence, the relevant price to determine the marginal cost of utility at t is $\tilde{q}^e_t$. Likewise, if there's some amount of spot consumption at $t+1$, that's because the constraint the SBL is not binding: $\tilde{B} - B_t > 0$. That means that on the margin, a unit of savings today translates into spot consumption whose price is 1. Otherwise, if the SBL is binding and only chained goods are consumed at $t+1$, the relevant price is q_{t+1}. Hence, the cost to deliver marginal utility by saving, is $\tilde{q}^s_{t+1}$. This generalized Euler equation, is critical in the characterization that follows.

State Variable and Recursive Representation. The model admits a recursive representation in the state variable, $Z_t \in B_t \times \tilde{B}_{t+1} \times \tilde{B}_{t+1}$. Thus, for any variable m_t, with abuse of notation, I define a function, $m(\cdot) : [0, \bar{B}]^3 \to \mathbb{R}_+$ such that $m_t = m(Z_t)$. Under the recursive representation, from (9) and clearing in the asset market, the saver spot expenditures are given by:

$$C^s(B) = (1-\beta) B.$$

In turn, from (10) the worker spot are given given by:

$$S^w\left(B, \tilde{B}\right) = \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta) B\right\}.$$

and the worker chained expenditures obtained from the income-expenditures identity[12], (8)

$$\mu\left(B, \tilde{B}\right) = 1 - (1-\beta) B - \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta) B\right\}$$

The price of chained consumption, $q\left(B, \tilde{B}\right)$, is obtained from the inverse of TFP $\mathcal{A}^{-1}\left(\mu\left(B, \tilde{B}\right)\right)$. We obtain the equilibrium chained consumption from the relation, $X^w\left(B, \tilde{B}\right) = \mu\left(B, \tilde{B}\right)/q\left(B, \tilde{B}\right)$. Likewise, we can define $\tilde{q}^e$ and $\tilde{q}^s$, using (13) and the functions $S^w\left(B, \tilde{B}\right)$, $X^w\left(B, \tilde{B}\right)$, and $q\left(B, \tilde{B}\right)$. In turn, output and

[12]The formula satisfies $\mu\left(B, \tilde{B}\right) \in [0, 1 - (1-\beta) B]$.

TFP are respectively given by $\mathcal{Y}\left(\mu\left(B,\tilde{B}\right)\right)$ and $\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)$.

To finalize the recursive representation, recall that any recursive representation involves a map from the current state variable to the future state variables. The mapping for the exogenous states is given. For the endogenous state variable, we define the function $B'(Z)$ such that $B_{t+1} = B'(Z_t)$. From (9), we have that

$$B' = \beta R(Z) B,$$

where $R(Z)$ is the equilibrium rate of return. Thus, the final object we need to complete the full characterization is $R(Z)$. The following proposition presents the exact solution solution:

Proposition 5. (Equilibrium Rates and Expenditures): *In a recursive equilibrium $R(Z)$ is given by:*

$$R(Z) = \frac{1}{\beta}\gamma(Z)$$

where $\gamma(Z)$ solves:

$$\gamma = \frac{1 + \left(q\left(\gamma B, \tilde{B}'\right) - 1\right)\max\left\{\tilde{B}' - \gamma B, 0\right\}\mathbb{I}\left[\tilde{B}' < 1 + \beta\gamma B\right]}{1 + \left(q\left(B, \tilde{B}\right) - 1\right)\max\left\{\tilde{B} - B, 0\right\}\mathbb{I}\left[\tilde{B} < 1 + \beta B\right]}.$$

Moreover, $\gamma(Z)$ is the growth rate of debt.

- [TBA] Explanation goes here

Figure 6 plots the growth rate of debt, $\beta R(Z)$. We should note that there are for regions of relevance that depend on the cases in the function. The discontinuity arises due to the indicators. The solution is obtained by replacing the market clearing condition into the the worker's Euler equation.

Steady States. Before we analyze the transitional dynamics of this model, I characterize the set of possible steady states. In a steady state, the price q and the real rate R are fixed. I drop time subscripts and use ss to denote a steady state value. In principle, a steady state could feature chained expenditures, if $X_{ss}^{w} > 0$ and

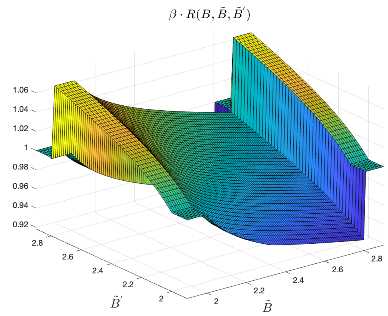

Figure 6: Equilibrium Rate.

Note: This figure plots $R(Z)$, fixing $B = 1$.

$q_{ss} < 1$. However, this turns out to be impossible, as the following Proposition demonstrates.

Proposition 6. *If intra-period borrowing is feasible $\tilde{B}_{ss} > 0$. Then, a steady state can only feature spot expenditures, $C^w_{ss} = S^w_{ss}$ and $q_{ss} = 1$. Moreover, the economy is in steady state at t if and only if the current debt level B_t satisfies:*

$$B_t \leq B^\star = \frac{1}{\beta}\left(\tilde{B}_{ss} - 1\right). \tag{14}$$

The proposition showcases as long as some intra-period debt borrowing is allowed, all steady-states are non-disrupted. For that to occur, condition (14) must hold. Let's provide some intuition on why this is the case. At steady state, from the saver's solution described in Proposition 3, we know that the economy can be in steady state only if $\beta R^b = 1$. Coupled with the worker's solution, 4 the worker's consumption can be in steady state if either $X > 0$ and $S = 0$ or $X = 0$ and $s > 0$. The former case can only occur if $\tilde{B}_{ss} = 0$. In turn, the latter case can only occur if the debt level of the worker is not high enough that it violated his borrowing debt limit, and this occurs when $B_t \leq B^\star$.

Now consider where the economy is at steady state, but a single individual worker has debt above $B^\star$. That worker will delever at a rate consistent with (4). As he delever, his consumption increases up to the point where $B_t \geq B^\star$. If all workers are above this level, the economy transition to steady state, and this has effects on the real interest and the implicit price q_t. An implication of this result, is that if there are any disruptions in the economy, these have to do with temporary low exogenous borrowing limits or temporarily high endogenous debt levels.

A Credit Crunch. We have observed that all steady states are non-disrupted, in the sense that there is no chained consumption. I now describe the transitions toward a steady state and, later, argue that they are inefficient. This the reason is that because cash-stripped households do not internalize that by shopping without cash, they lengthen the payments chains. They don't internalize that by saving a little, the following period they could all gain a marginal benefit. To do that, we first solve for the model's dynamics.

We now consider a sudden unexpected decline in the borrowing limit $\tilde{B}_t$. We start from $\tilde{B}_{ss}$, a steady state level. We let the limit fall to $\tilde{B}_0$ and the study a sequence that increases and converges back to steady state. The reversion speed can be geometric, or modeled as a one time jump.

Transitions toward steady state: A special case. To present a an analytic solutions, I begin with a violent transition. In a violent transition in which $\tilde{B}_t$ falls and remains depressed for $t \in \left[0, T^{crunch}\right]$ and then jumps back to steady state. In this case, debt remains constant:

Proposition 7. (Violent Transtions): *Consider a borrowing limit $\tilde{B}_{ss}$ and a steady-state value of debt B_{ss}. Also, consider a sequence $\left\{\tilde{B}_t\right\}$ of borrowing limits such that $\tilde{B}_t \leq B_{ss}$ for $t \in \left[0, T^{crunch}\right]$ and $\tilde{B}_t = \tilde{B}_{ss}$ for $t > T^{crunch}$. Then, $t \in \left[0, T^{crunch}\right]$, the worker maintains its steady–state expenditure level, $R_t = 1/\beta$, and debt remains constant. Moreover, the worker consumes only chained goods during $t \in \left[0, T^{crunch}\right]$ and only spot goods thereafter.*

This proposition showcases the possibility of extreme dynamics. There is no smoothing at all by the worker. Instead, the economy reacts passively and workers do not cutback chained expenditures. We now describe the transition when a partial recovery is expected.

Proposition 8. (Violent Transtions): *Consider a borrowing limit $\tilde{B}_{ss}$ and a steady-state value of debt B_{ss}. Also, consider a sequence $\left\{\tilde{B}_t\right\}$ of borrowing limits such that $\tilde{B}_t \leq B_{ss}$ for $t \in 0, T^{crunch},_t = \tilde{B}_{ss}$, for $t > T^{crunch}$ and $\tilde{B}_{T^{crunch}} \in \left[B_{ss}, \tilde{B}_{ss}\right]$. Then, $t \in \left[0, T^{crunch} - 2\right]$, the worker maintains its steady–state expenditure level, $R_t = R_{ss} = 1/\beta$, and debt remains constant. Moreover, the worker consumes only chained goods during $t \in \left[0, T^{crunch}\right]$ and only spot goods thereafter. At $T^{crunch} - 1$ the worker accumulates debt, $R_{T^{crunch}} > 1/\beta$, and X_t increases. At T^{crunch}, the worker repays debt, $R_t < 1/\beta$ and X_t decreases.*

- No hysteresis. No delay in returning to steady state allocations. No “over accumulation” of debt.
- Figure 7 presents two transitions corresponding to the cases we considered in the Propositions above. We can observe the pattern of over-accumulation and increase in chained consumption during an intermediate phase.

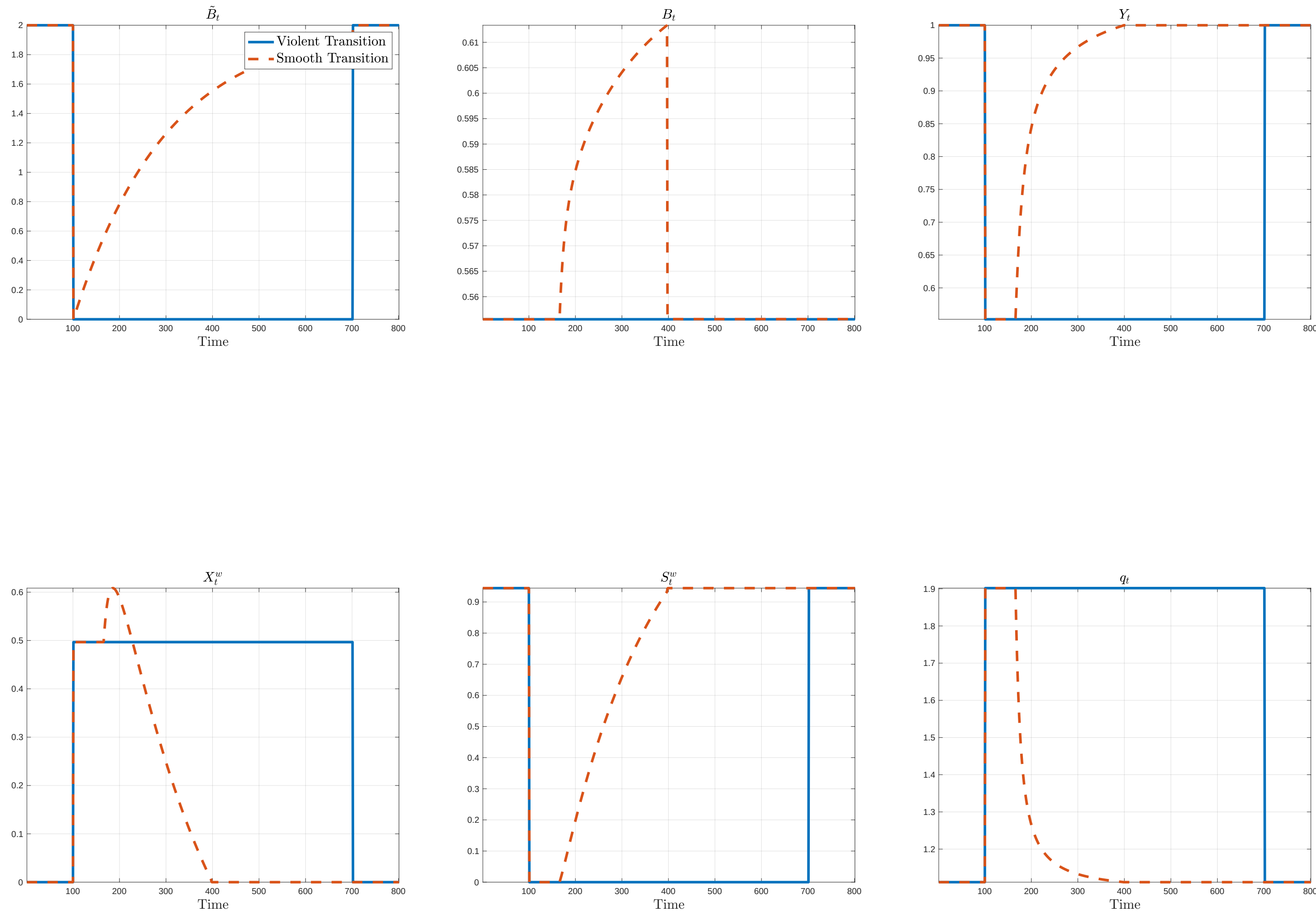

Figure 7: Transition after Credit Crunch.

Note: This figure reports a numerical example of a credit crunch episode. I consider two possibilities, I violent and a smooth transition.

Discussion: Borrowing vs Debt Limit. The distinction between borrowing and debt limits has technical and economic motivation: The technical motivation is that the borrowing limit allows us to study an unexpected credit crunch. Although an unexpected jump in the debt limit is not well-defined mathematically, an unexpected jump in the borrowing limit is.[13] In turn, the economic motivation is that if a bank wants to cut back on credit, it may be convenient to tighten the borrowing limit, but not necessarily to force households to repay debt principal immediately.[14]

4. Ex-Post Policies

Inefficiency. In Section 2 I alluded to the efficiency losses provoked when during payments-chain crises: as payments chains lengthen, chains get longer and production is delayed leading to losses in TFP. That description has to do with the transaction level which can be treated as part of the productive technology. Once expenditures are determined, there's nothing the government can do to eliminate the TFP losses because it does not have the technology to reorganize transactions. From a policy standpoint, what the government can do is influence the consumption and savings decisions that ultimately lead to the inefficient use of productive resources. The rest of this section is devoted to fleshing out the inefficiencies that follow from the consumption-savings decisions.

Next, I will show that both, static consumption decisions are inefficient and debt accumulation is inefficient. That is, I will show that relative to a market equilibrium, a planner would like to distort distort consumption decisions and alter the real rate of return on savings. Regarding static consumption decisions, there are two sides to the externality. On one hand, workers may consume too much chained consumption in a given period of time, because they don't internalize

[13]With an unexpected change in the debt limit, there would be a positive mass of households violating their debt limits. This does not apply to the borrowing limit $\tilde{s}_t$. An alternative approach is to study a gradual shock to debt limits as in **?**.

[14]When a bank extends a loan, it increases its liabilities. This is not true about a loan rollover. During crises, banks may want to roll over debt, although they are unwilling to extend loans because the latter consumes regulatory capital. In addition, if loan repayment is suddenly forced, it can trigger default which may lead to costly underwritings.

their effect of their consumption on the average chain length—an production of other. On the other hand, the wealthy consume to little.

Finally, the real interest rate may be excessively high or low relative to the debt dynamics that a planner would like to induce. Unfortunately, as we will see, the policy correction required is draconian. The planner will reduce the consumption of workers during a credit crunch, even though these are the agents that are hurt the most. It does so, promising to reduce the future interest rate, something that reduces their debt burdens.

A Ramsey Problem. We consider now a various Ramsey planners that differ in their available set of instruments, but respect the debt constraints in the model. Naturally, if we endow the Ramsey planner with transfers, then it is obvious it can circumvent any credit constraints: it could simply tax everyone and distribute consumption tokens in the desired consumption amount.

We first consider a sequence of credit taxes $\left\{\tau_t^k\right\}$, labor taxes $\left\{\tau_t^\ell\right\}$, and consumption taxes $\{\tau_t^c\}$. Let B_{ss} be a steady-state level of debt and θ a Pareto weight associated with that level of debt. The Pareto weights consistent with a given steady-state satisfy:

$$\frac{1-\beta B_{ss}}{\beta_{ss} B_{ss}} = \frac{\theta}{1-\theta}$$

We treat B_{ss} and θ as inputs to the planner problems. The first Ramsey problem is:

Problem 3. (Ramsey Problem):*Taking $B_0 = B_{ss}$ as given and a sequence of borrowing limits $\left\{\tilde{B}_t\right\}$, the Ramsey Planner maximizes:*

$$\max_{\left\{\tau_t^k,\tau_t^c,\tau_t^\ell\right\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the saver's budget constraint and optimality conditions:

$$\left(1-\tau_t^k\right)\frac{B_{t+1}}{R_{t+1}} + \left(1+\tau_t^c\right)C_t^s = B_t,\ \forall t \geq \mathbf{0}$$

$$\frac{C_{t+1}^{s}}{C_{t}^{s}}=\beta\left[\frac{1+\tau_{t}^{c}}{1+\tau_{t+1}^{c}}\right]\left(1-\tau_{t}^{k}\right)R_{t+1},\ \forall t\geq \mathit{0}$$

and the workers's constraints and optimality conditions:

$$B_{t}+\left(1+\tau_{t}^{c}\right)\left(S_{t}^{w}+q_{t}X_{t}^{w}\right)=\frac{B_{t+1}}{R_{t+1}}+1-\tau_{t}^{\ell},\ \forall t\geq \mathit{0}$$

$$\frac{C_{t+1}^{w}}{C_{t}^{w}}\equiv\beta\left[\frac{1+\tau_{t}^{c}}{1+\tau_{t+1}^{c}}\right]\left[\frac{1+\left(q_{t}-1\right)\mathbb{I}_{[X_{t}>0]}}{1+\left(q_{t+1}-1\right)\mathbb{I}_{[S_{t+1}=0]}}\right]R_{t+1},\ \forall t\geq \mathit{0}$$

$$C_{t}^{w}=S_{t}^{w}+X_{t}^{w}$$

$$S_{t}=\min\left\{\max\left\{\tilde{B}_{t}-B_{t},0\right\},C_{t}^{w}\right\}\ \textit{and}\ X_{t}=C_{t}^{w}-\min\left\{\max\left\{\tilde{B}_{t}-B_{t},0\right\},C_{t}^{w}\right\},\ \forall t\geq \mathit{0}$$

and respecting the payments constraints:

$$\mu_{t}=q_{t}X_{t}^{w}.$$

and the implicit cost of chained consumption $q_{t}=\mathcal{A}\left(\mu_{t}\right)^{-1}$ *and the budget balance constraint:*

$$\tau_{t}^{k}\frac{B_{t+1}}{R_{t+1}}+\tau_{t}^{c}\left(C_{t}^{s}+C_{t}^{w}\right)+\tau_{t}^{\ell}=0,\ \forall t\geq \mathit{0}.$$

The Ramsey planner distorts the economy with credit, labor, and consumption taxes, in order to avoid the externalities. The planner takes into account the optimality conditions of the agent behavior, their constraints and their market clearing conditions. It chooses taxes subject to a budget balance condition. Naturally, there's no role for government expenditures with complete instruments. We now turn to a primal planner problem, one where the planner can chose directly consumption of agents and choses an accounting variable, B_t, that determines dynamic constraints.

- If we endow the planner with only a credit tax and a transfer to workers, then consumption rule of savers is undistorted, and the interest rate absorbs the effect. Interest rate changes are akin to a transfer to savers.

- Therefore, the cleanest exercise is to introduce a time-varying uniform credit tax together with consumption taxes.

Now the problem that respects the condition.

Problem 4. (Primal I):

Taking $B_0 = B_{ss}$ and the time zero borrowing limit $\left\{\tilde{B}_t\right\}_{t\geq}$ as given, the primal Ramsey Planner maximizes:

$$\max_{\{C_t^s, X_t^w, B_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the resource constraint:

$$1 = C_t^s + S_t + \mathcal{A}\left(\mu_t\right)^{-1} X_t, \ \forall t \geq \mathbf{0}$$

$$\mu_t = \mathcal{A}\left(\mu_t\right)^{-1} X_t^w, \ \forall t \geq \mathbf{0}$$

$$S_t = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\} \textit{ and } X_t = C_t^w - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}, \ t = 0.$$

$$C_t^s \leq B_t$$

Notice that the constraint set in the Primal Ramsey problem includes the constraints of the original problem. This is immediate since market clearing in the asset market and the budget balance, implies, by Walras's law that the resource constraint holds, and naturally, only the time zero credit limit is imposed.

- There are two types of constraints. Static constraints and dynamic constraints.
  - Dynamics regards the choice of B_t.
- Static constraints regard the mix between C_t^s and X_t, given the level of S_t possible for our given construction.

The following lemma demonstrates that the solution to the primal problem coincides with the solution to original problem.

Lemma 1. (Implementability Conditions):

- *Assume that a solution to the primal problem is such that $C_t^s < B_t$. Then, the solution $\{C_t^*, X_t^*, B_t^*\}$ to the Primal problem is the solution the the Ramsey* *problem with credit taxes and consumption taxes. The sequence of taxes that implements the solution is:*

- *Conditions go hear.*

- The proof is obtained by noticing that there exists a path of $\left\{\tau_t^k, \tau_t^c, \tau_t^l\right\}$ that produces a path of consumption and B_t, consistent with the solution in the primal problem.

A special case of this problem occurs when it is possible to construct a path for B_t such that there's chained consumption only at $t = 0$. This special case is interesting, because it tells us the nature of the correction in consumption, in isolation of the path of B_t.

We have the following Lemma which shows that the solution to the original Ramsey problem coincides with the primal problem.

Proposition 9. (Solution to the Primal Problem II.):

The credit crunch in the primal problem lasts one period and the economy returns to steady-state immediately. Moreover, the planner distorts consumption at time zero where:

$$S_0 = \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}\right\}$$

and $\{X_0, C_0^s, \mu_0\}$ *solves:*

$$\frac{C_0^s}{S_0 + X_0} = \frac{(1-\theta)}{\theta}\left(\frac{q(\mu_0) - q'(\mu_0)\mu_0^2 + q'(\mu_0)\mu_0}{1 - q'(\mu_0)\mu_0^2}\right)$$

$$1 = C_0^s + S_0 + \mathcal{A}(\mu_0)^{-1} X_0$$

where:

$$\mu_0 = \frac{X_0}{C_0^s + S_0 + X_0}.$$

The proposition tells us that consumption is only biased in the first period of the crunch. After that, consumption can be carried out exclusively in a spot fashion. The intuition is that the credit tax has an influence on the rate of return on bonds, thus, it can distort wealth toward workers.

Proposition 10. (Insufficient consumption.): *Relative to the market solution, the solution to the Ramsey problem features more spot consumption.*

Credit Tax: tilting the evolution of debt. We now consider a problem where the planner cannot distort the consumption of savers because $\tau_t^c = 0$. Hence, we consider a sequence of credit taxes $\{\tau_t^k\}$ and the labor tax $\{\tau_t^l\}$.As an intermediate step, we begin presenting primal version of this problem.

Lemma 2. (Equivalence II): *The solution to the Ramsey problem without consumption taxes satisfies the following:*

I. *It induces the same path of debt as in the following primal problem:*

Taking $B_0 = B_{ss}$ *as given and a sequence of borrowing limits* $\left\{\tilde{B}_t\right\}$*, the Ramsey Planner maximizes:*

$$\max_{\left\{\hat{R}_{t+1}\right\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left((1-\beta)B_t\right) + \theta \log\left(\mathcal{A}\left(\mu_t\right)\mu_t + S_t^w\right)\right],$$

subject to the saver's budget constraint and optimality conditions

$$B_{t+1} = \hat{R}_{t+1}\beta B_t$$

and the resource constraint:

$$1 = \mu_t + S_t^w + (1-\beta)B_t$$

$$S_t \leq \max\left\{\tilde{B}_t - B_t, 0\right\}, \ \forall t \geq 0.$$

II. *The solution* $\left\{\hat{R}_{t+1}, B_{t+1}\right\}_{t\geq 0}$ *to the relaxed problem is implemented by the following sequence of credit and labor taxes:*

$$\left(1-\tau_t^k\right) = \frac{1}{\beta}\cdot\frac{\mu\left(B_{t+t},\tilde{B}_{t+1}\right) + S\left(B_{t+1},\tilde{B}_{t+1}\right)}{\mu\left(B_t,\tilde{B}_t\right) + S\left(B_t,\tilde{B}_t\right)}\cdot\frac{1+\left(q\left(\mu\left(B_t,\tilde{B}_t\right)\right)-1\right)\mathbb{I}_{\left[\mu\left(B_t,\tilde{B}_t\right)>0\right]}}{1+\left(q\left(\mu\left(B_{t+1},\tilde{B}_{t+1}\right)\right)-1\right)\mathbb{I}_{\left[S\left(B_{t+1},\tilde{B}_{t+1}\right)=0\right]}}.$$

Next, we present the solution to the primal problem.

Proposition 11. (Solution with Credit Tax Policy):

At $t=0$, the allocations coincide with the allocations of the competitive equilibrium without taxes. For any $t>0$, the choice of $\hat{R}_{t+1}$ that B_t satisfies:

$$\frac{1-(1-\beta)\,B_t}{(1-\beta)\,B_t}=\frac{\theta}{1-\theta}\cdot\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(\mu(B_t,\tilde{B}_t)\right)\right)\Sigma\left(B_t,B_t\right)$$

where $\Sigma\left(B,B\right)$ measures the change in the average price the worker consumption bundle relative to total consumption.

Hence, when $\tilde{B}_t\geq 1+\beta B_{ss}$, $B_t=B_{ss}$ and

$$\frac{1-(1-\beta)\,B_t}{(1-\beta)\,B_t}=\frac{\theta}{1-\theta}\cdot\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(\mu(B_t,\tilde{B}_t)\right)\right)$$

if $B_t<\tilde{B}_t$.

- Add figure for static solution

Transfers: uses matters.

- Transfers is not as useful as the consumption unless used for spending. We assumed labor subsidy does not enter the borrowing limit. If used to pay debt, its not that good. Connect with Richard Koo's discussion.

[Here I explain that it matters if the transfer is used for transactions or to pay debt.]

Government Spending: Pay for stuff vs. Spending. We now consider the optimality of government expenditures in this environment. We consider to possibilities: the case where government expenditures are financed with current tax receipts or future tax receipts. We begin with the latter problem.

Problem 5. (Spot Government Expenditures): *Taking $B_0=B_{ss}$ as given and a sequence of borrowing limits $\left\{\tilde{B}_t\right\}$, the Ramsey Planner maximizes:*

$$\max_{\left\{\tau_t^k,G_t\right\}_{t\geq 0}}\sum_{t\geq 0}\beta^t\left[(1-\theta)\log\left(C_t^s\right)+\theta\log\left(C_t^w\right)\right],$$

subject to the saver's budget constraint and optimality conditions:

$$\frac{B_{t+1}}{\left(1-\tau_t^k\right)R_{t+1}}+C_t^s=B_t,\ \forall t\geq 0$$

$$\frac{C_{t+1}^s}{C_t^s}=\beta\left(1-\tau_t^k\right)R_{t+1},\ \forall t\geq 0$$

and the workers's constraints and optimality conditions:

$$B_t+S_t^w+q_tX_t^w=\frac{B_{t+1}}{R_{t+1}}+1,\ \forall t\geq 0$$

$$\frac{C_{t+1}^w}{C_t^w}\equiv\beta\left[\frac{1+(q_t-1)\,\mathbb{I}_{[X_t>0]}}{1+(q_{t+1}-1)\,\mathbb{I}_{[S_{t+1}=0]}}\right]R_{t+1},\ \forall t\geq 0$$

$$C_t^w=S_t^w+X_t^w$$

$$S_t=\min\left\{\max\left\{\tilde{B}_t-B_t,0\right\},C_t^w\right\}\ \textit{and}\ X_t=C_t^w-\min\left\{\max\left\{\tilde{B}_t-B_t,0\right\},C_t^w\right\},\ \forall t\geq 0$$

and respecting the payments constraints:

$$\mu_t=\frac{X_t^w}{C_t^s+S_t^w+G_t+X_t^w}.$$

and the implicit cost of chained consumption $q_t=\mathcal{A}\left(\mu_t\right)^{-1}$ *and the budget balance constraint:*

$$\tau_t^kR_{t+1}=G_t,\ \forall t\geq 0.$$

We now study the case where G_t is consumed by the government in chained consumption. This requires expenditures to occur before tax collections.

Problem 6. (Spot Government Expenditures): *Taking* $B_0=B_{ss}$ *as given and a sequence of borrowing limits* $\left\{\tilde{B}_t\right\}$*, the Ramsey Planner maximizes:*

$$\max_{\left\{\tau_t^k,G_t\right\}_{t\geq 0}}\sum_{t\geq 0}\beta^t\left[(1-\theta)\log\left(C_t^s\right)+\theta\log\left(C_t^w\right)\right],$$

subject to the saver's budget constraint and optimality conditions:

$$\frac{B_{t+1}}{\left(1-\tau_t^k\right)R_{t+1}}+C_t^s=B_t,\ \forall t\geq 0$$

$$\frac{C_{t+1}^s}{C_t^s} = \beta\left(1-\tau_t^k\right)R_{t+1},\ \forall t \geq 0$$

and the workers's constraints and optimality conditions:

$$B_t + S_t^w + q_t X_t^w = \frac{B_{t+1}}{R_{t+1}} + 1,\ \forall t \geq 0$$

$$\frac{C_{t+1}^w}{C_t^w} \equiv \beta\left[\frac{1+(q_t-1)\,\mathbb{I}_{[X_t>0]}}{1+(q_{t+1}-1)\,\mathbb{I}_{[S_{t+1}=0]}}\right]R_{t+1},\ \forall t \geq 0$$

$$C_t^w = S_t^w + X_t^w$$

$$S_t = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\} \text{ and } X_t = C_t^w - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\},\ \forall t \geq 0$$

and respecting the payments constraints:

$$\mu_t = \frac{X_t^w + G_t}{C_t^s + S_t^w + G_t + X_t^w}.$$

and the implicit cost of chained consumption $q_t = \mathcal{A}\left(\mu_t\right)^{-1}$ *and the budget balance constraint:*

$$\tau_t^k R_{t+1} = G_t,\ \forall t \geq 0.$$

We have the following result:

Proposition 12. (Expenditure multipliers): *Consider the economy where government expenditures are spot. Then, the expenditures increase the value of the Ramsey problem:*

$$\underbrace{\frac{\theta}{S^w + X^w}}_{\text{marginal utility}} \underbrace{\frac{q(\mu) + q'(\mu)\,\mu\,(1-\mu)}{q'(\mu)\,\mu^2 - 1}}_{\text{net goverment multiplier}}$$

In turn, when government expenditures are chained,

$$\underbrace{\frac{\theta}{S^w + X^w}}_{\text{marginal utility}} \underbrace{\frac{q(\mu) + q'(\mu)\,\mu\,(1-\mu)}{q'(\mu)\,(1-\mu)\,\mu - 1}}_{\text{net goverment multiplier}}.$$

For $\mu \to 1$*, we have that the spot expenditure multiplier is:*

$$\frac{q(1)}{q'(1) - 1} > \frac{q(1)}{-1}.$$

and the chained expenditure multiplier:

$$\frac{q(1)}{-1} < 0$$

- clearly, since $q' > 0$, for small expenditures, there's a positive benefit only if expenditures are used to speed up payments.

5. Conclusion

There are many views about the nature of business-cycles. Some economists view shocks to the financial sector as episodes the catalyzer of subsequent economic downturns. Set aside that aspect, views differ once it comes to explaining the aftermath of the credit crunch that follows the disruptions in the financial sector. The new-Keynesian view, for example, articulates how societies waste resources if prices cannot adjust and aggregate demand remains deficient. The idea is particularly attractive because during financial crises, there is a generalized sense that that the economy suffers from coordination failures and demand externalities.

For monetarist economists, or any who believes that there is something inherently special about the financial crisis—*payments and money in a broad sense*—this description feels incomplete. Although monetarist economies share the view that policy can fix the coordination failures, they feel at ease that demand-driven theories are not explicit about payments or credit: there's nothing that makes financial crises special! Yet, if one thinks about it, the deepest and longest recessions of the past century have all been triggered by problems within the financial system. The Great Depression, the Japanese lost decades, the Great Recession that has affected the US and Europe for almost a decade now, all had some financial catalyst. If coordination failures don't have financial and payments nature, they put all recessions in the same bucket and policy recommendations are the same.

The contribution of this paper is to propose a payments interpretation financial crises and the failures that follow. The claim is that the core economic problem in the aftermath of a crises is not price rigidity, but an inefficient distribution of liquid funds. This inefficient distribution causes a delay in payments and is a form of coordination failures. Whereas the policy recommendations have a similar flavor to those that emerge in environments with price rigidity, the paper shows that those directed to circumvent the liquidity problem are the most effective.

In building the theory, I made two important shortcuts. First, I assumed that all transactions in the economy are of the same size. Second, I assumed that transactions take time to be executed when they are not spot transactions. Developing a general version of $\mathcal{A}$ that can handle a richer sequence of transactions and their corresponding delays is important to take the model to the data. Nonetheless, I believe the lessons from this paper are robust to such departures.

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Online Appendix

Appendix (Nor for publication)

A. Table of Contents

Contents

B. Proofs of Section 2

B.1 Proof of Proposition 2

Part 1. Derivation of TFP. I first derive the *expected* output generated by coupled expenditures in a given chain. We use this to find the expected output of a worker in a production relation. This will be equal to the expected value of the per-worker production in a n-length chain, taking expectation across n's. Notice that in the following augmented n-length chain—augmented by the single worker in the $n+2$ position—

$$\left\{s, \underbrace{x}_{1}, \underbrace{x}_{\delta}, \ldots, \underbrace{x}_{\delta^{n-1}}, \underbrace{s}_{\delta^{n}}\right\}$$

the production generated by workers in a production relation with n coupled shoppers is $\sum_{m=1}^{n} \delta^m$ as we do not consider the first worker for being related to a single shopper. Hence, the per-worker production in a n-length chain is

$$\bar{y}_n^x = \frac{1}{n} \sum_{m=1}^{n} \delta^m = \frac{\delta}{n} \left(\frac{1-\delta^n}{1-\delta} \right).$$

Now, recall that a couple will necessarily fall in a chain with length $n \geq 1$. Hence, the distribution of lengths for couples is $G(n;\mu)$ conditional on $n \geq 1$—since the first draw is a couple with probability μ. Thus, the distribution size of the payments chains among coupled expenditures is

$$G^x(n;\mu) = \frac{(1-\mu)\,\mu^n}{\mu},$$

where G^x denotes the distribution of lengths for chains with couples—note that it integrates to 1 as $n \to \infty$.

Next, we turn to our goal of finding the expected output of a worker in a production relation with at least one coupled expenditures:

$$\begin{aligned}\mathbb{E}\left[\bar{y}^x\right] &= \sum_{n=1}^{\infty} \bar{y}_n^x G^x\left(n;\mu\right),\\ &= \sum_{n=1}^{\infty} \frac{\left(1-\mu\right)\mu^n}{\mu}\cdot\frac{\delta}{n}\left(\frac{1-\delta^n}{1-\delta}\right),\\ &= \frac{\left(1-\mu\right)}{\mu}\cdot\frac{\delta}{\left(1-\delta\right)}\cdot\sum_{n=1}^{\infty}\left(\frac{\mu^n}{n}-\frac{\left(\delta\mu\right)^n}{n}\right),\\ &= \frac{\left(1-\mu\right)}{\mu}\cdot\frac{\delta}{\left(1-\delta\right)}\cdot\ln\left(\frac{1-\delta\mu}{1-\mu}\right),\end{aligned}$$

here the last equality comes from the fact that

$$\begin{aligned}\sum_{n=1}^{\infty} a^{n-1} = \frac{1}{1-a} \leftrightarrow\\ \sum_{n=1}^{\infty}\frac{a^n}{n} = \ln\left(\frac{1}{1-a}\right)\end{aligned}$$

for $|a| < 1$ because of the linearity of the derivative operator. We call $\mathcal{A}\left(\mu\right) = \mathcal{Y}^x\left(\mu\right) = \mathbb{E}\left[\bar{y}^x\right]$. Next, we derive expected output. The fraction of chains of length 1 is $\left(1-\mu\right)$. Production in these chains is 1 unit of output. The fraction of workers in couples is μ, and they produce on average $\mathcal{Y}^x\left(\mu\right)$. Thus, total output is:

$$\begin{aligned}\mathcal{Y}\left(\mu\right) &= \left(1-\mu\right)+\mu\mathcal{Y}^x\left(\mu\right)\\ &= \left(1-\mu\right)+\mu\frac{\left(1-\mu\right)}{\mu}\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\\ &= \left(1-\mu\right)\left(1+\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right).\end{aligned}$$

Next, we obtain the derivative and limits of $\mathcal{Y}\left(\mu\right), \mathcal{Y}^x\left(\mu\right)$.

Part 2. Limits. Limits as $\mu \to 0$. We first consider the limit of $\mathcal{A}$

$$\lim_{\mu\to 0}\mathcal{A}\left(\mu;\delta\right) = \frac{\delta}{\left(1-\delta\right)}\lim_{\mu\to 0}\left(\frac{1}{\mu}-1\right)\cdot\ln\left(\frac{1-\delta\mu}{1-\mu}\right) = \lim_{\mu\to 0}\frac{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\mu}.$$

The last term is the ratio of to variables that converge to zero.

Using L'Hospital's rule:

$$\lim_{\mu\to 0}\frac{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\mu}=\frac{\delta}{(1-\delta)}\frac{\lim_{\mu\to 0}\left(\frac{1}{1-\mu}-\frac{\delta}{1-\delta\mu}\right)}{1}=\delta.$$

where we used that:

$$\frac{\partial\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\partial\mu}=\frac{1-\mu}{1-\delta\mu}\left(\frac{1-\delta\mu}{1-\mu}\right)\left(\frac{1}{1-\mu}-\frac{\delta}{1-\delta\mu}\right)=\left(\frac{1}{1-\mu}-\frac{\delta}{1-\delta\mu}\right).$$

Next, we consider the limit of $\mathcal{Y}$

$$\lim_{\mu\to 0}\mathcal{Y}(\mu;\delta)=\lim_{\mu\to 0}(1-\mu)\lim_{\mu\to 0}\left(1+\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right)=1.$$

Limits as $\mu\to 1$. We first consider the limit of $\mathcal{A}$

$$\lim_{\mu\to 1}\mathcal{A}(\mu;\delta)=\frac{\delta}{(1-\delta)}\lim_{\mu\to 1}\left(\frac{1}{\mu}-1\right)\lim_{\mu\to 1}\ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

This is the product of a ratio that goes to 0 and a ratio that goes to infinity. Using L'Hospital's rule:

$$\lim_{\mu\to 1}\mathcal{A}(\mu;\delta)=\frac{\lim_{\mu\to 1}\left(-\frac{1}{\mu^2}\right)}{\lim_{\mu\to 1}\left(\frac{1}{1-\mu}-\frac{\delta}{1-\delta\mu}\right)}=0.$$

Next, we consider the limit of $\mathcal{Y}$

$$\lim_{\mu\to 1}\mathcal{Y}(\mu)=\lim_{\mu\to 1}(1-\mu)+\lim_{\mu\to 1}\mu\lim_{\mu\to 1}\mathcal{Y}^x(\mu)=0.$$

Limits as $\delta\to 0$. We first consider the limit of $\mathcal{A}$:

$$\lim_{\delta\to 0}\mathcal{A}(\mu;\delta)=\left(\frac{1}{\mu}-1\right)\lim_{\delta\to 0}\frac{\delta}{(1-\delta)}\cdot\lim_{\delta\to 0}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)=0.$$

Next, we consider the limit of $\mathcal{Y}$

$$\lim_{\delta\to 0}\mathcal{Y}(\mu)=(1-\mu)+\mu\lim_{\delta\to 0}\mathcal{Y}^x\mathcal{A}(\mu;\delta)=(1-\mu).$$

Limits as $\delta \to 1$. We first consider the limit of $\mathcal{A}$:

$$\lim_{\delta \to 1} \mathcal{A}(\mu;\delta) = \left(\frac{1}{\mu}-1\right) \lim_{\delta \to 1} \delta \cdot \lim_{\delta \to 1} \frac{1}{(1-\delta)} \cdot \lim_{\delta \to 1} \ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

This derivative is of the ratio of two numbers that goes to zero. Thus, we employ L'Hospital's rule again. We obtain:

$$\lim_{\delta \to 1} \mathcal{A}(\mu;\delta) = \left(\frac{1}{\mu}-1\right) \frac{\lim_{\delta \to 1} \frac{1-\mu}{1-\delta\mu}\left(\frac{-\mu}{1-\mu}\right)}{-1} = \left(\frac{1-\mu}{\mu}\right)\left(\frac{\mu}{1-\mu}\right) = 1.$$

Next, we consider the limit of $\mathcal{Y}$

$$\lim_{\delta \to 1} \mathcal{Y}(\mu) = (1-\mu) + \mu \lim_{\delta \to 1} \mathcal{Y}^{x} \mathcal{A}(\mu;\delta) = 1.$$

Thus, we have obtained all the limits of interest.

Part 3. Monotonicity. Next, we investigate the derivatives of $\mathcal{A}$ and $\mathcal{Y}$ and their concavity. Thus, we have that:

$$\mathcal{A}(\mu;\delta) = \left(\frac{1}{\mu}-1\right) \cdot \frac{\delta}{(1-\delta)} \cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

Thus, we obtain:

$$\mathcal{A}_\mu = \frac{\delta}{(1-\delta)}\left(\left(-\frac{1}{\mu^2}\right) \cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right) + \left(\frac{1}{\mu}-1\right)\left(\frac{-\delta}{1-\delta\mu} + \frac{1}{1-\mu}\right)\right).$$

Factoring out $-1/\mu^2$, we obtain:

$$\begin{aligned} \mathcal{A}_\mu &= -\frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu(1-\mu)\left(\frac{-\delta}{1-\delta\mu} + \frac{1}{1-\mu}\right)\right) \\ &= -\frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1/\mu-\delta}{1/\mu-1}\right) - \mu\left(\frac{-\delta(1-\mu)}{1-\delta\mu} + 1\right)\right) \\ &= -\frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right). \end{aligned}$$

To show that the function is indeed negative, we need to show that the term in the parenthesis is positive:

$$\ln(1-\delta\mu) - \left(\frac{\mu-\delta\mu}{1-\delta\mu}\right) > \ln(1-\mu).$$

A concave function must satisfy:

$$f(x) + f'(x)|y-x| > f(y).$$

Let

$$x = \delta\mu$$

and

$$y = \mu.$$

Then define:

$$f(x) \equiv \ln(1-x).$$

Then, since $x > y$ because $\{\delta, \mu\} < 1$, we have that:

$$|y-x| = (\mu - \delta\mu).$$

Thus, by concavity of the natural logarithm we have:

$$\ln(1-\delta\mu) - \underbrace{\left(\frac{\mu-\delta\mu}{1-\delta\mu}\right)}_{f'(x)|y-x|} > \ln(1-\mu).$$

which proves the desired inequality. Hence, we have showed that $\mathcal{A}_\mu < 0$.

Therefore, we also obtain that:

$$\mathcal{Y}_\mu = -1 + \mu\mathcal{A}_\mu = -\left(1 + \frac{\delta}{(1-\delta)}\frac{1}{\mu}\underbrace{\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right)}_{>0}\right).$$

Part 3. Concavity. Next we perform the convexity analysis. We have that:

$$\mathcal{A}_{\mu\mu} = \frac{\delta}{(1-\delta)}\left[2\frac{1}{\mu^3}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) - \frac{1}{\mu^2}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu} - \frac{1-\delta}{1-\delta\mu} - \mu\delta\frac{1-\delta}{(1-\delta\mu)^2}\right)\right.$$

We second term in parenthesis becomes:

$$\begin{aligned}
\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu} - \frac{1-\delta}{1-\delta\mu} - \mu\delta\frac{1-\delta}{(1-\delta\mu)^2} &= \left(\frac{1}{1-\mu} - \frac{(1-\delta\mu) - \delta\mu(1-\delta)}{(1-\delta\mu)^2}\right) \\
&= \left(\frac{1}{1-\mu} - \frac{1-2\delta\mu+\mu\delta^2}{(1-\delta\mu)^2}\right) \\
&= \left(\frac{1-2\delta\mu+(\delta\mu)^2-(1-\mu)(1-\mu\delta^2)}{(1-\mu)(1-\delta\mu)^2}\right) \\
&= \left(\frac{1-2\delta\mu+(\delta\mu)^2-1(1-\mu\delta^2)+\mu(1-\mu\delta^2)}{(1-\mu)(1-\delta\mu)^2}\right) \\
&= \left(\mu\frac{\delta^2-2\delta+1}{(1-\mu)(1-\delta\mu)^2}\right) \\
&= \mu\frac{(\delta-1)^2}{(1-\mu)(1-\delta\mu)^2}.
\end{aligned}$$

Thus, we have that:

$$\mathcal{A}_{\mu\mu} = \frac{\delta}{(1-\delta)}\frac{1}{\mu^3}\left[2\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - 2\mu\left(\frac{1-\delta}{1-\delta\mu}\right) - \frac{\mu^2(1-\delta)^2}{(1-\mu)(1-\delta\mu)^2}\right].$$

Then, we add the second and third terms to obtain:

$$-\mu\left(\frac{1-\delta}{1-\delta\mu}\right)\left(2+\frac{\mu^2(1-\delta)}{(1-\mu)(1-\delta\mu)}\right) = -\mu\left(\frac{1-\delta}{1-\delta\mu}\right)\left(\frac{2-\mu-3\delta\mu+2\delta\mu^2}{(1-\mu)(1-\delta\mu)}\right).$$

As a result, the second derivative is:t:

$$\mathcal{A}_{\mu\mu} = \frac{\delta}{(1-\delta)}\frac{1}{\mu^3}\left[\ln\left(\left(\frac{1-\delta\mu}{1-\mu}\right)^2\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\left(\frac{2-\mu-3\delta\mu+2\delta\mu^2}{(1-\mu)(1-\delta\mu)}\right)\right].$$

The function is strictly concave if:

$$\ln\left(\frac{1-\delta\mu}{1-\mu}\right)^2 - \mu(1-\delta)\frac{(2-\mu+\delta\mu(2\mu-3))}{1-\mu}\left(\frac{1}{1-\delta\mu}\right)^2 < 0$$

or

$$-\ln(1-\mu)^2 < -\ln(1-\delta\mu)^2 + \left(\frac{1}{1-\delta\mu}\right)^2 \left(\frac{\mu(1-\delta)}{(1-\mu)}\left(2-\mu+2\delta\mu^2-3\delta\mu\right)\right) \quad (15)$$

We observe that the function

$$f(x) \equiv -\ln x$$

is convex. Define

$$x = (1-\delta\mu)^2$$

and

$$y = (1-\mu)^2.$$

We perform some operations:

$$\begin{aligned} y-x &= (1-\mu)^2-(1-\delta\mu)^2 \\ &= \mu^2-2\mu(1-\delta)-(\delta\mu)^2 \\ &= \mu\left(\mu-2(1-\delta)-\delta^2\mu\right) \\ &= \mu\left(2(1-\delta)-\mu\left(1-\delta^2\right)\right) \\ &= (1-\delta)\mu(2-\mu(1+\delta)). \end{aligned}$$

Then:

$$\begin{aligned} (1-\delta)\mu(2-\mu(1+\delta)) &= \frac{(1-\delta)\mu}{(1-\mu)}(2(1-\mu)-(1-\mu)\mu(1+\delta)) \\ &= \frac{(1-\delta)\mu}{(1-\mu)} \cdot \left(2-3\mu-\delta\mu+\mu^2+\delta\mu^2\right) \end{aligned}$$

The term inside the parenthesis equals:

$$\begin{aligned} \left(2-2\mu-\left(\mu+\mu\delta-\mu^2(1+\delta)\right)\right) &= 2-3\mu-\mu\delta+\mu^2+\delta\mu^2 \\ &= \left(2-\mu-3\mu\delta+2\delta\mu^2\right)+\mu^2+2\mu+2\mu\delta-2\delta\mu^2 \\ &= \left(2-\mu-3\mu\delta+2\delta\mu^2\right)+\upsilon \end{aligned}$$

where

$$\upsilon = \mu^2 + 2\mu + 2\delta\mu\left(1-\mu^2\right) > 0.$$

Hence, we have that:

$$y - x = \frac{(1-\delta)\,\mu}{(1-\mu)} \cdot \left(2 - 3\mu - \delta\mu + \mu^2 + \delta\mu^2 + \upsilon\right).$$

By strict convexity of $-\ln x$, it must be the case that:

$$f(y) > f(x) + f'(x)\,|y-x|.$$

Since $f'(x) = -1/x$ but $x > y$, we have that:

$$f'(x)\,|y-x| = -\frac{1}{x}\,(y-x)\,.$$

Next, we replace the definitions and observe that:

$$\begin{aligned}
-\ln(1-\mu)^2 &< -\ln(1-\delta\mu)^2 - \left(\frac{1}{1-\delta\mu}\right)^2 (y-x) \\
&= -\ln(1-\delta\mu)^2 - \left(\frac{1}{1-\delta\mu}\right)^2 \left(\frac{\mu(1-\delta)}{(1-\mu)}\left(2-\mu+2\delta\mu^2-3\delta\mu\right)\right) \\
&= -\ln(1-\delta\mu)^2 - \left(\frac{1}{1-\delta\mu}\right)^2 \left(\frac{\mu(1-\delta)}{(1-\mu)}\left(\left(2-\mu-3\mu\delta+2\delta\mu^2\right)+\upsilon\right)\right) \\
&\leq -\ln(1-\delta\mu)^2 - \left(\frac{1}{1-\delta\mu}\right)^2 \left(\frac{\mu(1-\delta)}{(1-\mu)}\left(\left(2-\mu-3\mu\delta+2\delta\mu^2\right)\right)\right)
\end{aligned}$$

where the last inequality follows from $\upsilon > 0$. Thus, we have shown that (15) holds. Thus, $\mathcal{A}_{\mu\mu} < 0$ which guarantees the concavity of $\mathcal{A}$.

Next, we verify the convexity of output:

$$\begin{aligned}
\mathcal{Y}_{\mu\mu} &= \mathcal{A}_{\mu} + \mu\mathcal{A}_{\mu\mu} \\
&= -\frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) + \frac{\delta}{(1-\delta)}\frac{2}{\mu^2}\mu\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) \\
&\quad -\frac{\mu}{\mu^2}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu} - \frac{1-\delta}{1-\delta\mu} - \mu\delta\frac{1-\delta}{(1-\delta\mu)^2}\right) \\
&= \frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) - \frac{\mu}{\mu^2}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu} - \frac{1-\delta}{1-\delta\mu} - \mu\delta\frac{1-\delta}{(1-\delta\mu)^2}\right) \\
&= \mu\left(\frac{\delta}{(1-\delta)}\frac{2}{\mu^3}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) - \frac{\mu}{\mu^2}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu} - \frac{1-\delta}{1-\delta\mu} - \mu\delta\frac{1-\delta}{(1-\delta\mu)^2}\right.\right. \\
&\quad -\frac{\delta}{(1-\delta)}\frac{\mu}{\mu^3}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) \\
&= \mathcal{A}_{\mu\mu} - \frac{\delta}{(1-\delta)}\frac{\mu}{\mu^3}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) \\
&< 0.
\end{aligned}$$

Part 5. Inverse productivity.

Now, we study the inverse of productivity. Let

$$q(\mu;\delta) = \mathcal{A}^{-1}(\mu;\delta).$$

Clearly, the function has the limits:

$$\lim_{\mu\to 0} q(\mu;\delta) = \delta^{-1} \text{ and } \lim_{\mu\to 1} q(\mu;\delta) = \infty \text{ and } \lim_{\delta\to 0} q(\mu;\delta) = \infty \text{ and } \lim_{\delta\to 1} q(\mu;\delta) = 1.$$

We also have that:

$$q_{\mu} = -\frac{\mathcal{A}_{\mu}}{\mathcal{A}^2} > 0.$$

We use the limit of the derivative of this function:

$$q_{\mu}(\mu) = \frac{\frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right)}{\left(\frac{(1-\mu)}{\mu}\cdot\frac{\delta}{(1-\delta)}\cdot\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right)^2}$$

Next, we check the convexity of function:

$$q_{\mu\mu} = -\frac{\mathcal{A}_{\mu\mu}}{\mathcal{A}^2} + \frac{\mathcal{A}_{\mu}}{\mathcal{A}^3} > 0.$$

Hence, the price is a concave function.

Proof ends here.

C. Proofs of Section 3

C.1 Proof of Proposition 6

A steady state with both spot and chained consumption at any period is not possible since $\beta R = 1$ in any steady state. As a result, it is enough to proof that an all chained consumption steady state is not possible. Let's suppose that $c_{ss}^{w} = X > 0$ for all periods is a steady state solution of the workers' problem. At steady state I assume that $R_t = R$ for all $t \geq 0$ so we have

$$\max_{\{X_t, S_t, B_{t+1}\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log(c_t)$$
$$B_t + qX + \underbrace{S}_{=0} = \frac{B_{t+1}}{R} + 1$$

First, let's calculate a debt level that sustains the path of consumtion $\{(X, 0)\}_{t\geq 0}$. This is will make manipulations of the difference equation easy,

$$B_{ss} + qX = \frac{B_{ss}}{R} + 1$$
$$B_{ss} = \frac{R(1 - qX)}{(R - 1)}$$

This expression says that I can have a (positive) debt path (a constant one) as long as $qX < 1$ and the debt interest repayment is financed with the capitalized period savings. This is natural, in steady state, my per period consumption expenditure has to be lower than my real wage income. Let's first treat the case of $qX < 1$ and compute the debt path with a backward recursion

$$\begin{aligned} B_{t+1} &= RB_t - R(1-qX) \\ B_{t+1} &= R^{t+1}B_0 - \left(R^{t+1}-1\right) R\frac{(1-qX)}{R-1} \\ B_{t+1} - B_{ss} &= R^{t+1}(B_0 - B_{ss}) \end{aligned}$$

Case 1. If $B_0 - B_{ss} > 0$ we have forever increasing debt and this exceeds the natural debt limit at some finite time which will make impossible to consume X at that period in the future.

Case 2. If $B_0 - B_{ss} < 0$ then at some finite time $B_{t+1} = 0$ (at $R^{\tau+1}(B_0 - B_{ss}) = -B_{ss}$ and this necessarily happens at $\tau \geq 0$ because $B_0 > 0$). However, we only need a τ such that $0 < B_{\tau+1} < \tilde{B}$ (the spot borrowing limit) and this also happens at finite time since $B_0 > \tilde{B} > 0$. It happens at $\tau = \lceil j \rceil + 1$ where j is the time to close the intial gap and satisfies

$$\begin{aligned} B_{t+1} - \tilde{B} &= R^{j+1}(B_0 - B_{ss}) + B_{ss} - \tilde{B} = 0 \iff \\ \tilde{B} - B_{ss} &= R^{j+1}(B_0 - B_{ss}). \end{aligned}$$

In this case, in finite time (without the need of a deviation), the worker no longer has $(X, 0)$ as a solution because $S > 0$ will eventually become available and optimal.

Case 3. If $B_{ss} = B_0 > \tilde{B}$, the worker never changes the debt level and consuming $(X, 0)$ could be optimal. In this case, I cannot employ the argument above to show that this is not a steady state (because the feature before was that the debt level decreased due to the initial imbalance). However, we could use a deviation approach to show that there is an affordable and feasible plan, given prices $\{R\}_{t\geq 0}$ that achieves a higher lifetime utility. Suppose at time 0, the consumption is chosen $X - \varepsilon/q$ (for a fixed $\varepsilon > 0$) and later consumption is chosen X so we have the following equations for the path of debt

$$R + B_1 = RB_0 + RqX - R\varepsilon$$
$$R + B_{t+1} = RB_t + RqX, \quad \forall t \geq 1$$

solving backwards we have

$$\begin{aligned} B_{t+1} &= R^{t+1}B_0 - \left(R^{t+1} - 1\right) B_{ss} - \varepsilon R^{t+1} \\ B_{t+1} - B_{ss} &= -\varepsilon R^{t+1} \\ \tilde{B} - B_{t+1} &= \varepsilon R^{t+1} - \left(B_{ss} - \tilde{B}\right) \end{aligned}$$

and we observe that for finite time we can have B_{t+1} as low as we want. Suppose τ is the some (need not be the first) time such that $B_{\tau+1} < \tilde{B}$, The steady state plan has utility in periods 0 and $\tau + 1$

$$\log X + \beta^{\tau+1} \log X$$

my deviation plan has utility in periods 0 and $\tau + 1$

$$\log\left(X - \frac{\varepsilon}{q}\right) + \beta^{\tau+1} \log\left[X + \varepsilon R^{\tau+1} - \left(B_{ss} - \tilde{B}\right)\right]$$

so the change in utility from deviating is

$$\left\{\log\left(X - \frac{\varepsilon}{q}\right) - \log X\right\} + \beta^{\tau+1}\left\{\log\left[X + \varepsilon R^{\tau+1} - \left(B_{ss} - \tilde{B}\right)\right] - \log X\right\}$$
$$\log\left(1 - \frac{\varepsilon}{X} + \frac{\varepsilon}{X}\left(1 - \frac{1}{q}\right)\right) + \beta^{\tau+1} \log\left[1 + \frac{\varepsilon}{X} R^{\tau+1} - \frac{\left(B_{ss} - \tilde{B}\right)}{X}\right]$$

and by the mean value theorem (since log is continuous)

$$\log\left[1+\frac{\varepsilon}{X}R^{\tau+1}-\frac{\left(B_{ss}-\tilde{B}\right)}{X}\right] = \log\left[1+\frac{\varepsilon}{X}R^{\tau+1}\right]+\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\left[-\frac{\left(B_{ss}-\tilde{B}\right)}{X}\right]$$

$$\log\left[1-\frac{\varepsilon}{X}+\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)\right] = \log\left[1-\frac{\varepsilon}{X}\right]+\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\left[\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)\right]$$

with $\omega_1, \omega_2 \in (0,1)$, ω_1 depends on $\varepsilon R^{\tau+1}$ and ω_2 depends on ε only. Rearranging terms

$$\left\{\log\left[1-\frac{\varepsilon}{X}\right]+\beta^{\tau+1}\log\left[1+\frac{\varepsilon}{X}R^{\tau+1}\right]\right\}$$
$$+\left\{\underbrace{\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}_{\text{lower price benefit}}-\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\underbrace{\frac{\left(B_0-\tilde{B}\right)}{X}}_{\text{initial gap}}\right\}$$

We know that the first term can be made arbitrarily small choosing ε. Let's work with the second term. For each $\varepsilon > 0$ the "lower price benefit" term is fixed and the denominator of the slope of the "initial gap" is bounded below

$$1+\frac{\varepsilon}{X}R^{\tau+1}-\frac{\left(B_{ss}-\tilde{B}\right)}{X} < 1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}$$

and since the LHS of this inequality can get arbitrarily large with some large τ, the RHS too. Using this inequality I have

$$\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)-\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\frac{\left(B_{ss}-\tilde{B}\right)}{X}$$
$$> \frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)-\beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\frac{\left(B_{ss}-\tilde{B}\right)}{X}$$

where it is easy to see that (for any fixed ε) the quantity

$$\lim_{\tau+1} \beta^{\tau+1} \frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1} - \frac{\left(B_{ss}-\tilde{B}\right)}{X}} = 0$$

goes to zero. So given ε fixed, we can find $k > 0$ (sufficiently small) and τ depending on k satisfying

$$\underbrace{\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}_{\text{lower price benefit}} - \beta^{\tau+1}\frac{1}{1+\frac{\varepsilon}{X}R^{\tau+1}-\omega_1\frac{\left(B_{ss}-\tilde{B}\right)}{X}}\underbrace{\frac{\left(B_0-\tilde{B}\right)}{X}}_{\text{initial gap}} >$$

$$\underbrace{\frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}_{A(\varepsilon)=\text{lower price benefit}} - k > 0$$

So for every $\varepsilon > 0$ we can define k's positive but smaller than $A(\varepsilon)$ such that $A(\varepsilon) - k > 0$ and we can make this quantity as close as desired to $A(\varepsilon)$ (choosing a smaller $k > 0$). Now returning to the change of utility of the deviation, this expression can be made arbitrarily close to

$$\left\{\log\left[1-\frac{\varepsilon}{X}\right] + \beta^{\tau+1}\log\left[1+\frac{\varepsilon}{X}R^{\tau+1}\right]\right\} + A(\varepsilon)$$

As a consequence, it only remains to show that the above quantity is positive for small ε (fixing τ). Using a taylor expansion

$$\begin{aligned}
&-\left(\frac{\varepsilon}{X}\right)^2 - \beta^{\tau+1}\left(\frac{\varepsilon}{X}R^{\tau+1}\right)^2 + \mathcal{O}\left(\|\varepsilon\|^3\right) + A(\varepsilon)\\
&> -\left(\frac{\varepsilon}{X}\right)^2\left[1+R^{\tau+1}\right] + \frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)\\
&= a\varepsilon - b\varepsilon^2
\end{aligned}$$

because the third derivative adjustment for log is positive[15] and where

$$a = \frac{1}{1-\frac{\varepsilon}{X}+\omega_2\frac{\varepsilon}{X}\left(1-\frac{1}{q}\right)}\frac{1}{X}\left(1-\frac{1}{q}\right)>0, \quad b=\left[1+R^{\tau+1}\right]\frac{1}{X^2}>0$$

Finally, we know that a linear function (with positive coefficient) is always greater than a square (with positive coefficient) for $\varepsilon > 0$ small (because the square stays very close to zero for ε small). Namely, $a\varepsilon - b\varepsilon^2 > 0$. So the deviation was profitable, which finishes the proof of this case.

A special case happens when the initial gap is zero, i.e. $B_{ss} = B_0 = \tilde{B}$, but then we only need to wait one period after our deviation to increase spot consumption profitably. So a steady state solution with $qX < 1$ and $B_0 = B_{ss} = \tilde{B}$ is not possible.

Case 4. If $qX = 1$ then $B_{ss} = 0$ and only a zero initial level of debt is admissible. In this case, if $\tilde{B} > 0$ then spot consumption is available and because (from euler equation) $X > 0$, $S > 0$ is incompatible with a steady state for the saver, we get a contradiction. So an all chained consumption steady state with $qX = 1$ is not possible.

To seet that the economy is at steady state if and only if $B_t \leq B^\star = \frac{1}{\beta}\left(\tilde{B}_{ss}-1\right)$, we note that this is equivalent to $1-(1-\beta)B_t \leq \tilde{B} - B_t$ which is equivalent to being in steady state.

C.2 Proof of Propositions 3 and 4

Proof.

The proof of Proposition 3 is immediate. The proof of Proposition 4 requires more work. We begin with a perturbed version of Problem (xxx). Consider the following:

$$= \max .$$

Thus, we arrive at the recursion.

$$U'(c(B)) \equiv \frac{\beta R_{t+1}}{(q-1)\,\mathbb{I}_{\left[c(B)>\Xi\left(\tilde{B},B\right)^+\right]}+1}U'\left(c\left(RB-h+c(B)+(q-1)\left(c(B)-\Xi\left(\tilde{B},B\right)^+\right)^+\right)\right).$$

[15] $\left(-\frac{\varepsilon}{X}\right)^3+\beta^{\tau+1}\left(\frac{\varepsilon}{X}R^{\tau+1}\right)^3 \to \left(\frac{\varepsilon}{X}\right)^3\left[R^{2\tau+2}-1\right]>0$

Proof ends here.

C.3 Proof of Corollary xxx

Proof.

Recall that

$$c^r = (1-\beta)\,B$$

and the expenditures of workers is:

$$s + qx = h - (1-\beta)\,B.$$

Since we know that

$$s \leq \Xi\left(\tilde{B}, B\right)^+$$

we have that:

$$s = \min\left\{\Xi\left(\tilde{B}, B\right)^+, h - (1-\beta)\,B\right\}$$

$$x = \frac{h - (1-\beta)\,B - \min\left\{\Xi\left(\tilde{B}, B\right)^+, h - (1-\beta)\,B\right\}}{q}.$$

Then,

$$\begin{aligned}
\mu &= \frac{x}{c^r + s + x} \\
&= \frac{\frac{h-(1-\beta)B-\min\left\{\Xi\left(\tilde{B},B\right)^+,h-(1-\beta)B\right\}}{q}}{(1-\beta)\,B + \frac{h-(1-\beta)B-\min\left\{\Xi\left(\tilde{B},B\right)^+,h-(1-\beta)B\right\}}{q} + \min\left\{\Xi\left(\tilde{B}, B\right)^+, h-(1-\beta)\,B\right\}} \\
&= \frac{1}{1 + q\frac{(1-\beta)B+\min\left\{\Xi\left(\tilde{B},B\right)^+,h-(1-\beta)B\right\}}{h-(1-\beta)B-\min\left\{\Xi\left(\tilde{B},B\right)^+,h-(1-\beta)B\right\}}}.
\end{aligned}$$

With this we replace:

$$\mu = \frac{1}{1 + \frac{1}{\mathcal{A}(\mu)} \cdot \Gamma\left(B\right)},$$

where

$$\Gamma(B)=\frac{(1-\beta)B+\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}}{h-\left((1-\beta)B+\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}\right)}.$$

Then, using

$$A(\mu)=\frac{(1-\mu)}{\mu}\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)$$

we obtain:

$$1-\mu=\frac{\mu}{\mathcal{A}(\mu)}\cdot\Gamma(B).$$

- Could have solution if inverse is there. Cross-fingers!

We obtain a fixed point problem in μ. We call this object $\mu(B)$. And $q(B)=q(\mu(B))$.

- It may also be possible to solve for $q(B)$ or $\mu(B)$.

Next, we sum x and s in xxx to obtain:

$$c(B)=\frac{h-(1-\beta)B}{q(B)}+\left(1-\frac{1}{q(B)}\right)\min\left\{\Xi\left(\tilde{B},B\right)^{+},h-(1-\beta)B\right\}.$$

Now in the modified euler equation:

$$U'(c(B))=\frac{\beta R_{t+1}}{(q(\mu(B'))-1)\mathbb{I}_{\left[c(B)>\Xi\left(\tilde{B},B\right)^{+}\right]}+1}U'(c(B')).$$

Since we know $B'=\beta R_{t+1}B$ we obtain:

$$U'(c(B))=\frac{\beta R_{t+1}}{(q(\mu(\beta R_{t+1}B))-1)\mathbb{I}_{\left[c(B)>\Xi\left(\tilde{B},B\right)^{+}\right]}+1}U'(c(\beta R_{t+1}B)).$$

With this, we have one equation in one unknown, R.

- May be really simple for log
- The condition may hold everywhere, independent of t.

Proof ends here.

C.4 Proposition Condition...

Notice that the steady state is indeed a spot-transaction steady state, the price is $p = 1$. Assume that the worker consumes spot transactions forever. Then, the labor first-order condition is

$$h^{\nu} = 1.$$

Thus, there's a total of $h = 1$ labor effort. Thus,

$$Y = 1.$$

At steady state $R = 1/\beta$, otherwise the wealthy households will continue change their wealth. Then, there consumption is:

$$C^s = (1 - \beta) B.$$

By the clearing condition in the goods market,

$$C = 1 - (1 - \beta) B,$$

where the condition follows from the fact that all consumption is spot. Then,

$$s = 1 - (1 - \beta) B \leq \tilde{B} - B.$$

Hence, the condition follows. Finally, we must show that the worker does not wish to accumulate debt. Notice that he is would not accumulate debt even if the constraint were not binding ever. Thus, he must be at an optimum.

D. Proofs of Section ??

D.1 Proof of Proposition ??(regarding the transition after a special credit crunch)

Preliminary Observations. The economy starts from a given steady state ss^1. Thus, at the time of the shock, $t = 0$, $B_0 = B_{ss^1}$. From Proposition 3, we know that the optimal consumption

$$C_t^s = (1 - \beta) B_t, \ \forall t.$$

Assume that $\tilde{B}_0 > 0$ and that the sequence is increasing. Then, the worker consumes at least some amount by executing spot transactions. Namely,

$$S_0^w = \min \left\{ \tilde{B}_0 - B_0, C_0^w \right\}$$

and

$$X_0^w = \frac{C_0^w - \min\left\{\max\left\{\tilde{B}_0 - B_0, 0\right\}, C_0^w\right\}}{q_0}.$$

In particular, this happens at all periods:

$$S_t^w = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}$$

and

$$X_t^w = \frac{C_t^w - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}}{q_t}.$$

Now, we combine the household's expenditures

$$h = C_t^s + S_t^w + q_t X_t^w.$$

Thus, subbing (xxx) and (xxx) into (xxx), we obtain:

$$X_t^w = \frac{h - \left((1 - \beta) B_t + \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, C_t^w\right\}\right)}{q_t}.$$

Assume that the borrowing limit is binding such that X_t^w. Thus, we must have that:

$$X_t^w = \frac{h - \left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}{q_t} > 0.$$

Item (i). Combining (xxx) and the definition of μ, (xxx), we obtain:

$$\begin{aligned}
\mu &= \frac{\frac{h-\left((1-\beta)B_t+\max\left\{\tilde{B}_t-B_t,0\right\}\right)}{q_t}}{(1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\} + \frac{h-\left((1-\beta)B_t+\max\left\{\tilde{B}_t-B_t,0\right\}\right)}{q_t}} \\
&= \frac{h - \left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}{q_t\left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right) + h - \left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}.
\end{aligned}$$

Thus,

$$\frac{1}{\mu} = q_t \frac{\left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)}{h - \left((1-\beta)\,B_t + \max\left\{\tilde{B}_t - B_t, 0\right\}\right)} + 1.$$

Then, subtracting one from both sides, we obtain:

$$\frac{1-\mu}{\mu} \cdot \frac{1}{q_t} = \Lambda\left(B_t, \tilde{B}_t\right),$$

where

$$\Lambda\left(B, \tilde{B}\right) = \frac{\left((1-\beta)\,B + \max\left\{\tilde{B} - B, 0\right\}\right)}{h - \left((1-\beta)\,B + \max\left\{\tilde{B} - B, 0\right\}\right)}.$$

Then, observe that $q_t = 1/\mathcal{A}(\mu_t)$. Hence, we obtain:

$$\frac{1-\mu}{\mu} \cdot \frac{1}{q_t} = -(\mu_t).$$

where we applied the definition of $\mathcal{A}(\mu_t)$ in the second equality and use the following definition

$$-(\mu) \equiv \frac{1-\mu}{\mu}\mathcal{A}(\mu_t).$$

Thus, at any t where spot transaction are binding:

$$-(\mu) = \Lambda\left(B, \tilde{B}\right). \tag{16}$$

Notice that in any equilibrium, because B satisfies the natural debt limit, $h > (1-\beta)B$. Since chained expenditures are positive, it must be that $h - \left((1-\beta)B + \max\left\{\tilde{B} - B, 0\right.\right.$ is non-negative. Thus, $\Lambda\left(B, \tilde{B}\right) > 0$. Also, observe that $-(\mu)$ is decreasing in μ because by Proposition $\mathcal{A}(\mu_t)$, the term $\mathcal{A}$ is decreasing. Moreover, it has the limits: $\lim_{\mu\to 0} -(\mu) = \infty$ and $\lim_{\mu\to 1} -(\mu) = 0$. This means that there's a unique solution to (16) we denote by the function

$$\mu\left(B, \tilde{B}\right) \equiv \left\{\mu | -(\mu) = \Lambda\left(B, \tilde{B}\right)\right\}.$$

There is no known analytic root to this problem.

However, in equilibrium

$$\mu_t = \mu\left(B_t, \tilde{B}_t\right)$$

and moreover:

$$q\left(B, \tilde{B}\right) = \frac{\mu\left(B, \tilde{B}\right)}{\left(1 - \mu\left(B, \tilde{B}\right)\right)} \cdot \frac{(1-\delta)}{\delta} \cdot \ln\left(\frac{1 - \mu\left(B, \tilde{B}\right)}{1 - \mu\left(B, \tilde{B}\right)\delta}\right).$$

$$q_t = q\left(B_t, \tilde{B}_t\right).$$

Item (ii). Next, we combine the worker and savers first order condition.

Consider the first-order condition at a point where t is such that $X_t^w > 0$ and $S_t^w = 0$. Thus, we have that:

$$\begin{aligned} C_{t+1}^w &= X_t^w + S_t^w \\ &= \frac{h - (1-\beta)B_t - S_t^w}{q_t} + S_t^w \\ &= \frac{h - (1-\beta)B_t + (q_t - 1)S_t^w}{q_t} \\ &= \frac{h - (1-\beta)B_t + (q_t - 1)\max\left\{\tilde{B}_t - B_t, 0\right\}}{q_t}. \end{aligned}$$

Now consider the worker's first order condition [Important: must bind at t+1 also...else, must replace by future S, not, the binding one...just patch with X=0]

$$
\begin{aligned}
q_t \beta R_t &= \frac{C^w_{t+1}}{C^w_t} \\
&= \frac{q_t}{q_{t+1}} \frac{h-(1-\beta)\,B_{t+1}+(q_{t+1}-1)\max\left\{\tilde{B}_{t+1}-B_{t+1},0\right\}}{h-(1-\beta)\,B_t+(q_t-1)\max\left\{\tilde{B}_t-B_t,0\right\}}.
\end{aligned}
$$

Thus,

$$
\beta R_t = \frac{1}{q_{t+1}} \frac{h-(1-\beta)\,B_{t+1}+(q_{t+1}-1)\max\left\{\tilde{B}_{t+1}-B_{t+1},0\right\}}{h-(1-\beta)\,B_t+(q_t-1)\max\left\{\tilde{B}_t-B_t,0\right\}}.
$$

Now, recall that $B_{t+1}=\beta R_t B_t$. Thus, we obtain:

$$
\beta R_t = \frac{1}{q_{t+1}} \frac{h-(1-\beta)\,\beta R_t B_t+(q_{t+1}-1)\max\left\{\tilde{B}_{t+1}-\beta R_t B_t,0\right\}}{h-(1-\beta)\,B_t+(q_t-1)\max\left\{\tilde{B}_t-B_t,0\right\}}.
$$

Hence, we obtain:

$$
\beta R = \frac{1}{q\left(\beta RB,\tilde{B}'\right)} \frac{h-(1-\beta)\,\beta RB+\left(q\left(\beta RB,\tilde{B}'\right)-1\right)\max\left\{\tilde{B}-\beta RB,0\right\}}{h-(1-\beta)\,B+\left(q\left(B,\tilde{B}\right)-1\right)\max\left\{\tilde{B}-B,0\right\}}.
$$

This gives us an implicit solution which we can solve:

$$
R\left(B,\tilde{B},\tilde{B}'\right)
$$

and

$$
B'\left(B,\tilde{B},\tilde{B}'\right)=\beta R\left(B,\tilde{B},\tilde{B}'\right)B.
$$

E. Proofs of Section 4