

Purchase history and product personalization*

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Abstract

Product personalization opens the door to price discrimination. A rich product line allows for higher consumer satisfaction, but the mere choice of a product carries valuable information about the consumer that the firm can leverage for price discrimination. Controlling the degree of product personalization provides the firm with an additional tool to curb ratcheting forces arising from consumers' awareness of being price discriminated. Indeed, a firm's inability to not engage in price discrimination introduces a novel distortion: The firm offers a subset of the products that it would offer if, instead, the firm could commit to not price discriminate. Doing so gives commitment power to the firm: By 'pooling' consumers with different tastes to the same variety the firm commits not to learn their tastes.

KEYWORDS: product-line design, price discrimination, dynamic mechanism design, information design, limited commitment

JEL CLASSIFICATION: D84, D86, L12, L13, L15

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1 Introduction

Repeated interactions between firms and consumers are ubiquitous. Consider, for instance, a consumer who purchases repeatedly from an online retailer, such as Amazon. Repeated interactions also occur in more traditional settings, where the purchase of a basic good is often followed by an add-on: Car dealers sell vehicles and maintenance plans, insurance companies sell home insurance and various “umbrella” insurance add-ons, printer manufacturers sell printers and ink cartridges. These repeated interactions are not limited to consumers and firms: In government contracting, once a firm is hired to do a big project, smaller side projects often follow.

Particularly in online settings, firms nowadays have an immense capacity to store information that can be used both for product personalization and for future price discrimination. Product personalization allows the firm to better meet its customers’ needs, thereby increasing consumer welfare. However, via their product choices, consumers reveal information about their preferences and willingness to pay which, in turn, allows firms to engage in price discrimination later on, that is, to charge different customers different prices for identical products produced at the same marginal cost.¹ Indeed, this trade-off between the value of product personalization and the perils of price discrimination is of concern to policy makers:

The increased availability of behavioral data has also encouraged a shift from third-degree price discrimination based on broad demographic categories towards personalized pricing. [...] Much of this activity facilitates personalized tracking and targeting, which create value by helping firms better identify and serve consumers’ needs.²

Consumers are well aware of such price discriminating practices and change their behavior by making it harder for firms to use the information, if the cost of being charged a higher price does not justify the benefit of receiving a bespoke product (e.g., by

¹Thus, we define price discrimination as in Stigler (1966) and Varian (1989). Note this definition excludes, for instance, the model of Mussa and Rosen (1978) as a model of price discrimination: In Mussa and Rosen (1978), consumers pay different prices for goods of different quality.

²See Executive Office of the President of the United States (2015).

deleting their browser's cookies or refraining from revealing purchases).³ These *ratcheting* forces are strongest when consumers are sophisticated, demanding that the firms compensate them up front for the future rents that they give up when they reveal their information.

This forward-looking behavior by consumers is the reason the vast literature on intertemporal price discrimination, starting with the seminal work of Stokey (1979), shows that firms optimally choose not to price discriminate (see also Riley and Zeckhauser, 1983, Baron and Besanko, 1984, Wilson, 1993, and the references in the surveys of Varian, 1989 and Armstrong, 2006). The suboptimality of price discrimination, however, depends on the assumption that the firm can commit not to use the information gleaned from consumers' choices. Instead, without such commitment, price discrimination obtains (see, e.g., Hart and Tirole, 1988; Acquisti and Varian, 2005). It follows that price discrimination erodes the firm's profits, rather than increasing them.

Whereas firms use consumers' information both for product personalization and price discrimination, most of the literature has focused on the latter. This ignores a fundamental trade-off: A "rich" product line improves consumers' experience, but also makes purchase decisions more informative. Purchase histories become more detailed, thus allowing for even finer discrimination. In turn, a firm that can design its product line has an additional tool to prevent price discrimination: By selecting which products the consumer can choose from, the firm can control how much it learns about the consumer.

In this paper, we study how limited commitment (and hence, the temptation to price discriminate) shapes the design of a firm's product line. We answer this question in a stylized setting where a firm and a consumer interact over two periods and two different transactions. In the first period, the firm chooses its product line as in Mussa

³Indeed, in a report on the impact of big data on differential pricing, Executive Office of the President of the United States (2015) states

In particular, three broad trends suggest that concerns about big data and personalized pricing are not stifling consumer activity on the Internet. Those trends are: (1) the rapid growth of electronic commerce, (2) the proliferation of consumer-empowering technologies, and (3) the slow uptake of privacy tools.

and Rosen (1978). That is, it produces a good of varying quality at nondecreasing marginal cost. Furthermore, as in Mussa and Rosen (1978), tailoring quality to the consumer's tastes is beneficial: Consumers with higher types value higher quality more than consumers with lower types. In the second period, the firm sells an indivisible good, produced at zero marginal cost, as in the setting of Hart and Tirole (1988). In line with our motivation, one can think of the first period transaction as the basic good (i.e., the vehicle, the insurance plan, or the printer), whereas the second transaction is the add-on (i.e., the maintenance plan, insurance add-on, ink cartridge). The first period good captures the benefits to both the firm and the consumer from utilizing the consumer's information for product personalization. Instead, the second period product gives the firm the opportunity to use the information gleaned from personalization to price discriminate.

We assume that the consumer's valuation for each of the goods is her private information. When the consumer contracts with the firm in period 1, she does not know her valuation for the good in period 2. Instead, the consumer's value in period 1 is informative about her valuation in period 2: A consumer who values quality more is more likely to value the second period good more. To distinguish the consumer's private information in period 1 from that in period 2, we denote the consumer's valuation for the good in period 1 her *type*, and we reserve the term *valuation* for the value of the period 2 good. Whereas the consumer's period-1 type is drawn from a continuum, we assume that the consumer's period-2 valuation is binary.

As a benchmark, we derive the solution for the case in which the firm can commit at the beginning of the interaction to the mechanism that the consumer will face. Because we are analyzing a dynamic mechanism design problem, where the consumer's information evolves over time as in Pavan et al. (2014), the firm could have an incentive to intertemporally price discriminate.⁴ To make the comparison with limited commitment the starkest, we consider a setting in which intertemporal price discrimination is not

⁴While the literature on dynamic mechanism design with commitment continues to grow (see Bergemann and Välimäki, 2019 and the references therein), to the best of our knowledge only Deb and Said (2015) study optimal dynamic mechanism design under limited commitment. Whereas their model features persistent types and evolving population, ours features evolving private information within a fixed population.

optimal under commitment. In the optimal mechanism, the firm charges a price for the period 2 indivisible good that is *independent* of the quality purchased in period 1. This, in turn, implies the optimal mechanism in period 1 reduces to that in Mussa and Rosen (1978), which has an important implication. Whereas in Mussa and Rosen (1978), the firm distorts downwards the quality that each consumer type gets relative to what each type would get under symmetric information (the first best), interestingly, the product *line* that the firm chooses is the same as in the first best (see, e.g, Anderson and Celik, 2015). That is, in terms of the product line, the second best derived in Mussa and Rosen (1978) is observationally indistinguishable from the case in which the firm knows the consumer's information.

We then characterize the firm's optimal mechanism under limited commitment. Under limited commitment, the second period mechanism must be optimal given the information that the firm has at that point so it will necessarily depend on the purchase history. In particular, we show that the firm optimally chooses to *prune* the product line relative to the optimal mechanism under commitment (and hence, the first best). This choice, in turn, induces further distortions in the quality that each consumer type gets in the first period relative to the commitment solution. These distortions reflect the tension between the benefits of the use of the information, through product personalization, and the costs of its use, through its effect on the pricing scheme in period 2. By curtailing the range of products it offers the consumer, the firm obfuscates how much information it learns in period 1, and hence limits the amount of price discrimination in period 2.

Preview of results In our model, the firm's choice of a product line in period 1 depends on whether the firm would prefer to sustain low or high prices in period 2 under the commitment solution. Figures 1 and 2 depict the product line (left) and the choice of quality by each consumer type (right) in the first best (black), second best (blue), and under limited commitment (red) for each of these two cases. (Because the product line is the same under the first and second best, the left panel of each figure only depicts the product line for the first best and limited commitment.)

Figure 1 illustrates the optimal product line for the case in which the firm would

prefer to set a high price in period 2 under the commitment solution. Because higher consumer types in period 1 are associated to higher consumer valuations in period 2, in order to sustain a high price in period 2, the quality chosen by the consumer in period 1 must convince the firm that the consumer's type is high. The firm accomplishes this by curtailing the quality range at the bottom of the product line, forcing a larger range of consumer types to purchase the good of the lowest quality in period 1. The firm then can justify setting a high price in period 2, even after observing a consumer who bought a low quality product in period 1. In this case, the effect of limited commitment is detrimental both to the firm and to the consumer. By curtailing the quality range in period 1, more consumer types purchase the lowest quality good than in the commitment solution. Moreover, it charges higher prices to those consumers that purchase higher quality goods: After all, fewer substitutes are available in period 1 for these consumer types, and hence, the firm can charge a higher price than in the commitment solution. Despite the higher prices in period 1, however, the firm is worse off.

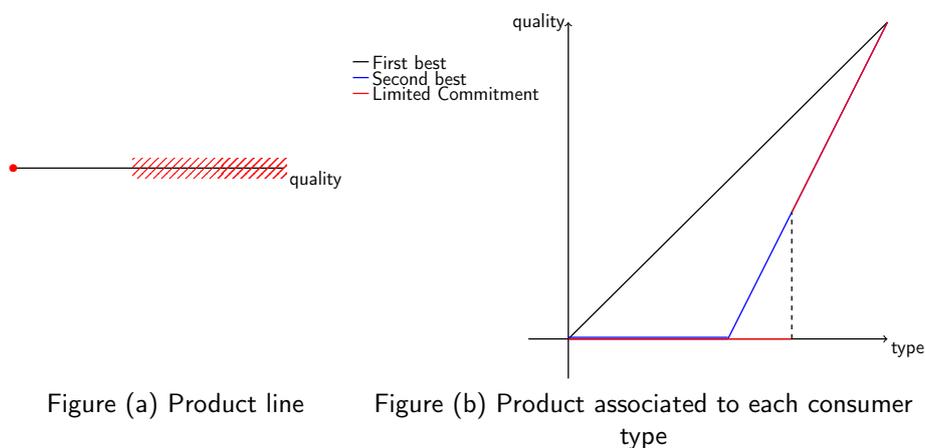


Figure 1: Product line (left) and product choice (right).

Figure 2 illustrates the optimal product line for the case in which the firm would prefer to set a low price in period 2 under the commitment solution. Because higher consumer types in period 1 are associated with higher valuations for the good in period 2, in order to sustain a low price, the quality chosen by a consumer in period 1 must convince the firm that the consumer's type is low. The firm accomplishes this by offering less

qualities in the mid-range to increase the set of period 1 consumers who will face a low price in period 2. However, the firm continues to offer high consumer types the same quality that they would have received under the commitment solution. This, in turn, implies that the firm price discriminates by charging higher prices to those consumers who buy at the high end of the product line. Indeed, as our analysis shows, the cost of failing to offer a personalized product to high consumer types in period 1 is not compensated by the benefit of avoiding price discrimination.

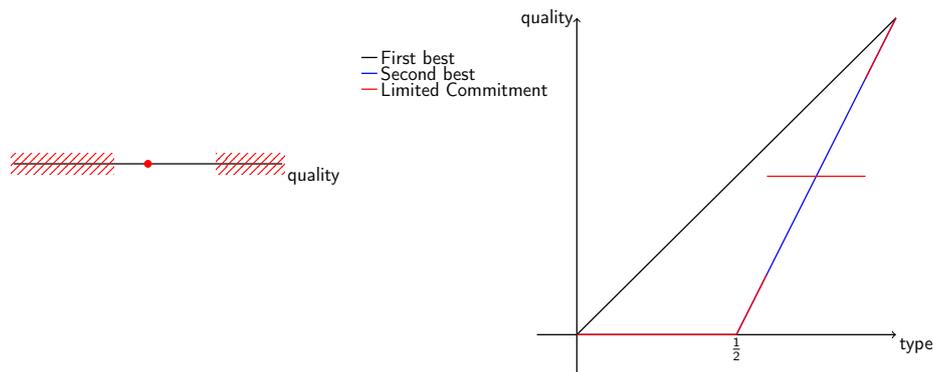


Figure (a) Product line Figure (b) Product associated to each consumer type

Figure 2: Product line (left) and product choice (right).

Our characterization highlights the trade-off that arises under limited commitment between the benefits of the use of information in period 1 to better cater to the consumer's period 1 valuation and the costs of doing so via its impact on the mechanism offered in period 2. The results in Propositions 2 and 3 show how the firm's sequential rationality constraints lead to allocative distortions above and beyond those due to informational rents, which include both under- and over-provision of quality in period 1. Underlying this rich pattern of distortions is the firm's attempt to obfuscate how much it learns about the consumer's private information, by sorting the consumer's types into (multiple) pooling and separation intervals. Interestingly, however, the optimal mechanism can be implemented with the firm observing the consumer choosing from a set of menus.

For reasons similar to those in Laffont and Tirole (1988), to derive the optimal mechanism under limited commitment, we cannot rely on the revelation principle (see, e.g.,

Myerson, 1986). Instead, we rely on Theorem 2 in Doval and Skreta (2020), which provides a revelation principle for limited commitment in Markov environments, like the one we consider in this paper.⁵ As we explain in Section 4, Theorem 2 in Doval and Skreta (2020) allows us to reduce the search for the firm's optimal mechanism to a constrained optimization program. In particular, by leveraging the analysis in Doval and Skreta (2020), we construct the firm's optimal product line by marrying elements of mechanism design and information design. On the mechanism design side, we leverage the first order approach in dynamic mechanism design and dynamic public finance to characterize the solution to a relaxed problem and then provide conditions under which the firm can implement the solution to the relaxed problem (see Pavan et al., 2014; Stantcheva, 2020). On the information design side, we leverage the techniques of information design for continuum type spaces to transform the design of the product line, and hence how much information the firm learns about the consumer, into an information design problem (e.g., Gentzkow and Kamenica, 2016; Kolotilin, 2018; Dworzak and Martini, 2019; Arieli et al., 2019).

Related Literature: Our work contributes to the literatures on product-line design (e.g., Mussa and Rosen, 1978; Itoh, 1983) and price discrimination, which for the most part have proceeded on separate tracks. An exception is Sun (2014), who studies a repeated version of the model in Mussa and Rosen (1978). Sun (2014) shows that offering a single variety may be optimal with binary values. Furthermore, when the firm chooses from a restricted class of mechanisms, he provides conditions under which a single variety is offered in the first period when types are drawn from a continuum: either it is optimal to offer a single variety under commitment, or the firm is patient so that it sacrifices product personalization today, in lieu of product personalization tomorrow. While our paper shares with Sun (2014) the observation that limited commitment limits varieties in the market, the results are not related otherwise: we do not restrict the set of mechanisms the firm offers to the consumer, but there is no product

⁵The terminology follows Pavan et al. (2014), who denote by Markov environments settings where (i) the consumer's private information follows a possibly nonhomogeneous Markov process, (ii) the principal and the consumer's payoffs are time-separable, and their flow payoffs depend only on today's allocation and the consumer's current type, and (iii) the transition probability may depend both on today's type and today's allocation.

personalization in period 2. Thus, even though there is no discounting across periods in our model, the firm still finds it optimal to offer fully personalized products in period 1 to consumers in the high-end of the type distribution.

Johnson and Myatt (2003) show that when two firms compete with differentiated products, product line pruning may be optimal because it softens competition by generating local monopolies. Villas-Boas (2004) examines the role of communication costs of different varieties, whereas Fudenberg and Tirole (1998) and Ellison and Fudenberg (2000) look at the intertemporal effects of product versioning. Kamenica (2008), and, more recently, Xu and Dukes (2019), examine to what extent firms can increase profits through an appropriate product-line design that leverages cognitive and behavioral biases of consumers. Our paper contributes to this strand by identifying a new rationale for product line pruning and studying the feedback between product personalization and price discrimination.

By considering the feedback between product personalization and price discrimination, we contribute to the long and influential literature on price discrimination (e.g., Riley and Zeckhauser, 1983; Armstrong, 2006) and especially to the works that study intertemporal price discrimination (e.g., Stokey, 1979; Hart and Tirole, 1988; Acquisti and Varian, 2005; Fudenberg and Villas-Boas, 2006). Because in our model the firm also designs its product line, the firm can condition on richer purchase histories when engaging in price discrimination.⁶

One of the costs of firms using consumers' information is the resulting loss in privacy. Thus, we relate to the works that study consumer privacy starting from the classic contributions of Taylor (2004) and Calzolari and Pavan (2006b).⁷ More recently, Bonatti and Cisternas (2020) and Argenziano and Bonatti (2020) also study data links across firms to determine pricing decisions, whereas Eilat et al. (2020) consider the design of

⁶Whereas in Johnson and Myatt (2003) product line pruning softens competition by increasing differentiation, Zhang (2011) shows that competition can also soften differentiation when price discrimination is possible. In Zhang (2011), two firms choose a location in a Hotelling line (a product) in the first period, anticipating that in the second period they can make price offers conditional on whether the consumer purchased from the firm or the rival. Zhang (2011) shows that both firms choose the same location in period one, making the decision to purchase uninformative.

⁷Cummings et al. (2015) study the impact of ad targeting in monopoly pricing in a two-period model with a continuum of types in the first period and binary types in the second period.

mechanisms subject to a privacy constraint. In a model with exogenous varieties, Ichihashi (2020) studies the one-shot interaction between a multi-product firm, that can send product recommendations, and a consumer, who can reveal information about her preferences. Relatedly, Hidir and Vellodi (2020) study incentive compatible market segmentations in a static market where the firm produces a horizontally differentiated product.⁸ Finally, Ali et al. (2019) uncover a reason consumers may prefer to reveal information to a firm, even if the firm produces a homogeneous product: It may allow the firm to serve consumers that it would have otherwise excluded.

Our model also relates to the literature on downstream markets, because the period-1 mechanism affects the “downstream” interaction between the firm and the consumer (see, e.g., Zheng, 2002, Calzolari and Pavan, 2006a, Calzolari and Pavan, 2006b, and more recently, Dworzak, 2020). Unlike in those papers, the upstream and downstream firm are not independent players in our model and hence, the downstream firm learns whatever information the upstream firm learns about the consumer’s type in period 1. From a technical perspective, this difference prevents us from invoking the results in Calzolari and Pavan (2006b) to argue that the commitment solution features no price discrimination.

From a methodological perspective, our paper contributes to the literature on mechanism design with limited commitment, by considering a setting where (i) the consumer’s private information in the first period is drawn from a continuum, and (ii) whereas the consumer’s private information is positively correlated across periods, it is not fully persistent. This is possible thanks to the results in Doval and Skreta (2020), who provide a revelation principle for environments such as those analyzed in this paper. Indeed, previous versions of the revelation principle for limited commitment only hold for environments with fully persistent and discrete types (see Bester and Strausz, 2001, 2007). Furthermore, the analysis in the papers that consider models with a continuum of types, such as Skreta (2006), Deb and Said (2015), Skreta (2015), does not readily extend to settings beyond those they consider.

⁸Note that these models do not feature price discrimination as we have defined it: Consumers pay different prices in these models because they buy different products.

Organization: The rest of the paper is organized as follows. Section 2 describes the model and notation. Section 3 solves two benchmarks: Section 3.1 considers the artificial case in which the firm only sells the indivisible good in period 2, whereas Section 3.2 characterizes the optimal mechanism under commitment. Section 4 derives the optimal mechanisms when the firm has limited commitment. Section 5 concludes. All proofs are in Appendix A.

2 Model

A profit-maximizing firm and a consumer interact over two periods, $t \in \{1, 2\}$. We assume the firm and the consumer are fully patient and thus, do not discount payoffs across periods.

In period 1, the firm produces a good of variable quality at a nondecreasing marginal cost. Thus, letting q_1 and x_1 denote the good's quality and the payment from the consumer to the firm in period 1, period-1 allocations are given by $(q_1, x_1) \in [0, \bar{Q}] \times \mathbb{R} \equiv A_1$. The firm incurs cost $c(q_1) = cq_1^2/2$ to produce quality q_1 . Note that the term *allocation* refers to both the quality and the transfer, that is, to the pair (q_1, x_1) .

In period 2, the firm produces an indivisible good at 0 marginal cost. Period-2 allocations are described by $(q_2, x_2) \in \{0, 1\} \times \mathbb{R} \equiv A_2$, where q_2 denotes whether the period-2 good is assigned to the consumer and x_2 denotes the payment from the consumer to the firm.

The consumer's valuation for each of the goods is private information. In period 1, if the consumer purchases a good of quality q_1 and pays x_1 , her flow payoff is $u_1(q_1, x_1, \theta) = \theta q_1 - x_1$, where $\theta \in [0, 1] \equiv \Theta$ denotes the consumer's type. In period 2, if she purchases the good and pay x_2 , their flow payoff is $u_2(q_2, x_2, v) = vq_2 - x_2$, where $v \in \{v_L, v_H\}$, $0 < v_L < v_H$.

We assume that the consumer's type in period 1 is distributed uniformly on $[0, 1]$, that is, $\theta \sim U[0, 1]$. In period 1, the consumer does not know her valuation for the good in period 2. Conditional on the consumer's type in period 1 being θ , her valuation

in period 2 is v_H with probability $p(\theta) = \theta$.⁹ In what follows, it is sometimes more informative to derive expressions without replacing the parametric assumptions so we reserve F_1 to denote the firm's belief about the consumer's type in period 1 and $p(\theta)$ to denote the probability that a consumer of type θ has value v_H in period 2.

3 Two benchmarks

Section 3 solves two problems that help build intuition for the optimal mechanism under limited commitment. **Section 3.1** considers a fictitious setting where the firm only designs the optimal mechanism to sell the period 2 good. The analysis in **Section 3.1** allows us to understand how beliefs about θ affect pricing decisions in period 2, which is useful to understand the optimal mechanisms under commitment and limited commitment. **Section 3.2** then characterizes the firm's optimal mechanism under the assumption of commitment.

3.1 Period-2 pricing without product-line design

Suppose the firm could only sell the indivisible good to the consumer, so that their interaction is limited to period 2. Standard arguments imply the optimal mechanism is then a posted price. Whether this posted price is v_L or v_H depends on the likelihood the firm assigns to the consumer's value being v_H . This likelihood, in turn, depends on the firm's beliefs in period 2 about the consumer's type, θ .

Letting F_2 denote the firm's belief in period 2 about θ , the firm assigns probability $\mathbb{E}_{F_2}[\theta] \equiv \mu_{F_2}$ to the consumer's valuation being v_H . Then, the optimal price in period 2 is given by:

⁹As the analysis that follows will make clear, the main role of the parametric assumptions is to enable the application of the existing tools of information design with a continuum type spaces, which have been developed exclusively for the case in which the sender and the receiver care only about the posterior mean (e.g., Gentzkow and Kamenica, 2016; Kolotilin, 2018; Dworczak and Martini, 2019; Arieli et al., 2019). However, the economic force underlying the distortions in the product line is more primitive and, as we discuss in **Section 5**, we expect that it will arise under more general assumptions.

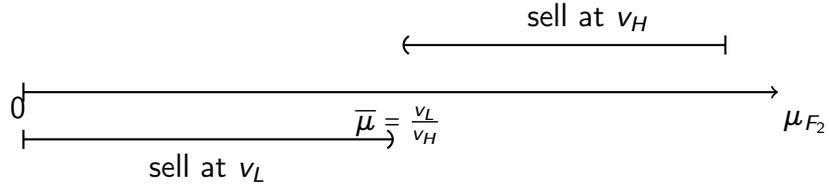


Figure 3: Optimal period-2 price as a function of μ_{F_2}

where $\bar{\mu} = v_L/v_H$ is the belief about v_H at which the firm is indifferent between selling at a price of v_H (obtaining revenue $\mu_{F_2} v_H$) and selling at a price of v_L (obtaining revenue of v_L).

Figure 3 illustrates two important themes for what follows. First, optimal period-2 pricing is sensitive to the information about θ , which gives rise to the possibility of price discrimination. Second, optimal period-2 pricing only depends on the posterior mean of θ , μ_{F_2} .

3.2 Product-line design under commitment

As our next benchmark, we consider the case in which the firm has full commitment. Our model is a special case of the environments studied in Pavan et al. (2014), so we can rely on the revelation principle to characterize the firm's optimal mechanism (see, e.g., Myerson, 1986).

A direct revelation mechanism consists of a tuple

$$\{(q_1(\theta), x_1(\theta), (q_2(\theta, v), x_2(\theta, v))_{v \in \{v_L, v_H\}}) : \theta \in [0, 1]\},$$

which specifies the allocation that the consumer receives in each period, as a function of the reports in each period. Importantly, when the consumer submits a type report, θ' , in period 1, she restricts the menu from which she chooses in period 2, to $(q_2(\theta', \cdot), x_2(\theta', \cdot))$.

The direct revelation mechanism must satisfy the following incentive compatibility constraints. In period 2, for all reported types θ' in period 1, the consumer must find

it optimal to reveal her value,¹⁰

$$u_2(\theta', v) \equiv vq_2(\theta', v) - x_2(\theta', v) \geq vq_2(\theta', v') - x_2(\theta', v'). \quad (\text{F-IC}_{\theta', v, v'})$$

Furthermore, in period 1, the consumer must find it optimal to reveal her type. That is, letting

$$U_1(\theta', \theta) \equiv \theta q_1(\theta') - x_1(\theta') + p(\theta)(v_H q_2(\theta', v_H) - x_2(\theta', v_H)) + (1 - p(\theta))(v_L q_2(\theta', v_L) - x_2(\theta', v_L))$$

denote the expected payoff of a type θ consumer who reports θ' in period 1, it must be the case that for all $\theta \in [0, 1]$ and all reports $\theta' \in [0, 1]$,

$$U_1(\theta, \theta) \geq U_1(\theta', \theta). \quad (\text{F-IC}_{\theta, \theta'})$$

Whereas the consumer could deviate by first misreporting θ and then misreporting the value in period 2, this choice is not optimal by (F-IC_{θ', v, v'}). Finally, the consumer must find it optimal to participate in the mechanism; that is, for all $\theta \in [0, 1]$,

$$U_1(\theta, \theta) \geq 0. \quad (\text{F-PC}_\theta)$$

The optimal mechanism under commitment then solves

$$\max_{q_1, x_1, q_2, x_2} \int_{\Theta} [x_1(\theta) - c(q_1(\theta)) + p(\theta)x_2(\theta, v_H) + (1 - p(\theta))x_2(\theta, v_L)] F_1(d\theta) \quad (\text{F-OPT})$$

subject to the constraints, (F-PC_θ), (F-IC_{θ, θ'}), (F-IC_{θ', v, v'}).

Proposition 1 describes the optimal mechanism and Figure 4 below illustrates it:

Proposition 1. *Suppose the firm has full commitment. Then, the optimal mechanism is as follows:*

¹⁰In the equation labels, the letter F denotes *full* commitment.

1. In period 1, a consumer with type θ obtains quality

$$q_1(\theta) = \begin{cases} 0 & \text{if } \theta < \frac{1}{2} \\ \frac{2\theta-1}{c} & \text{if } \theta \geq \frac{1}{2} \end{cases} .$$

2. In period 2, a consumer with type θ and valuation v obtains the following allocation. If $\bar{\mu} \leq 1/2$,

$$(q_2(\theta, v), x_2(\theta, v)) = \begin{cases} (1, v_H) & \text{if } v = v_H \\ (0, 0) & \text{otherwise} \end{cases} .$$

Instead, if $\bar{\mu} > 1/2$,

$$(q_2(\theta, v), x_2(\theta, v)) = (1, v_L) \text{ for } v \in \{v_L, v_H\}.$$

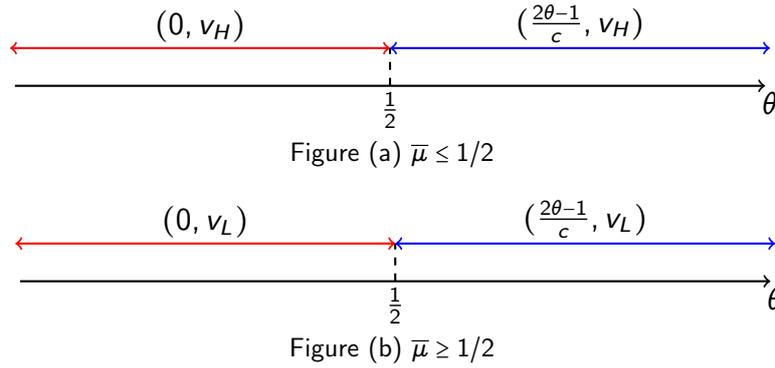


Figure 4: Period-1 quality and period-2 price under commitment as a function of θ .

See [Section A.1](#) for the proof. In what follows, we provide intuition for the optimal mechanism under commitment.

A feature of the optimal mechanism under commitment is the lack of price discrimination in period 2: All consumer types in period 1 face the same price in period 2 *regardless* of the quality purchased in period 1. The reason is that because the firm chooses the mechanism in period 1, the firm internalizes the information externalities across periods. To see why, note the following. The incentive compatibility constraint

in period 2 ($F-IC_{\theta',v,v'}$) implies the firm needs to leave rents to the consumer so that she truthfully reveals her value in period 2. However, the firm contracts with the consumer in period 1 so it can recoup part of these rents. Indeed, from the perspective of period 1, what matters for rents is (i) how θ determines the willingness to pay for quality (period-1 rents) and (ii) how informative θ is about the period-2 value v (portion of the period-2 rents that go to the consumer). That is, when a consumer with type θ reports θ' , her information rents are given by:

$$(\theta - \theta') (q_1(\theta') + u_2(\theta', v_H) - u_2(\theta', v_L)),$$

where the dependence of u_2 on θ' represents the possibility of price discrimination in period 2 as a function of the type report in period 1. Under our parametric assumptions, all consumer types are equally informative about v at the margin, and hence, price discriminating as a function of θ is not worthwhile for the firm.

The comparison between the pricing decisions in the optimal mechanism under commitment with those in [Figure 3](#) reveals that under commitment the firm uses the prior mean of θ to decide on the period 2 price. (We show this result formally in [Section 4](#); see [Figure 6](#).) Indeed, the threshold of $1/2$ against which $\bar{\mu}$ is compared is precisely the mean of θ at the prior. Thus, when the firm assigns ex-ante a high probability to the consumer's period-2 value being v_H (i.e., $\bar{\mu} < 1/2$), prices are high in period 2. Instead, when the firm assigns ex-ante a low probability to the consumer's period-2 value being v_L (i.e., $1/2 < \bar{\mu}$), prices are low in period 2.

Note the optimal mechanism under commitment fails to be sequentially rational. To see this, note that for consumer types above $1/2$, the quality provided at the end of period 1 fully reveals the consumer's private information. However, the price in period 2 is *independent* of this information. For instance, consider the case in which $\bar{\mu} > 1/2$ and a consumer type θ above $\bar{\mu}$. For such a consumer, the optimal period-2 price is v_H . However, under commitment, the firm can ignore the information revealed by the mechanism and instead, set a price of v_L .

4 Product-line design under limited commitment

Section 4 presents the firm's optimal mechanism under limited commitment. We start by formally introducing the game between the firm and the consumer and the solution concept.

Timing and strategies: At the beginning of each period $t \in \{1, 2\}$, the firm proposes a mechanism, \mathbf{M}_t . A mechanism \mathbf{M}_t consists of a set of input messages, M_t a set of output messages S_t , and a device $\varphi_t : M \mapsto \Delta(S_t \times A_t)$, which assigns to each input message $m \in M_t$ a distribution over output messages and allocations.

Observing the mechanism, the consumer chooses to accept or reject the mechanism. If the consumer rejects, she gets nothing from the firm and also makes no payments. That is, the allocation at the end of period t is $(q_t, x_t) = (0, 0)$. Instead, if the consumer participates, she *privately* submits an input message to the mechanism ($r_t \in \Delta(M_t)$). This message determines the distribution $\varphi(\cdot|m)$ from which the output message and the allocation are drawn. Both the firm and the consumer observe the output message and the allocation. The game ends at the end of period 2.

Figure 5 summarizes the timing of the game:¹¹

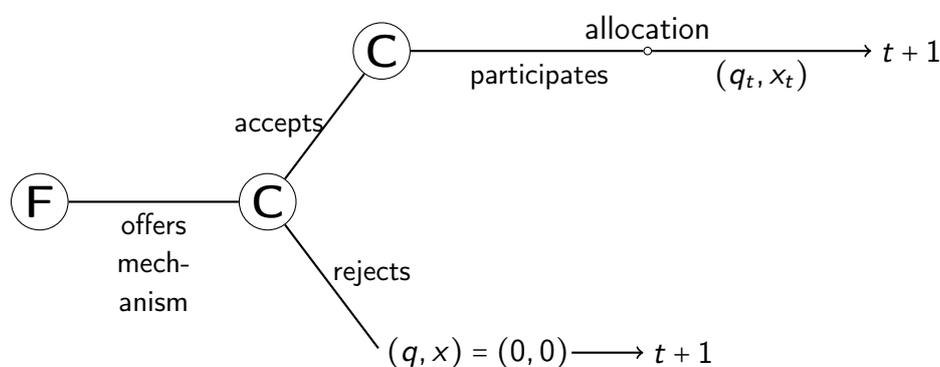


Figure 5: Timing with limited commitment

¹¹Under the described timing, the consumer may purchase the good in period 2 even if she did not purchase the good in period 1. Our analysis would remain the same if instead we added the restriction that the consumer must contract with the firm in period 1 to purchase the good in period 2, as in the add-on application.

Solution concept: We are interested in the best *perfect Bayesian equilibrium* (PBE) outcome for the firm. At a PBE, the firm's and the consumer's strategies are sequentially rational. Moreover, the firm's beliefs about the consumer's type are determined by Bayes' rule *where possible*.

To characterize the optimal mechanism under limited commitment, we proceed in two steps. First, we apply Theorem 2 in Doval and Skreta (2020). This allows us to restrict the search for the firm's optimal PBE to a well-defined set of mechanisms for the firm and to simple strategies for the consumer. Second, we solve for the firm's optimal mechanism in Section 4.1.

Revelation principle: Theorem 2 in Doval and Skreta (2020) implies that without loss of generality, the firm can be restricted to choosing in each period a mechanism such that the following hold. First, the set of input and output messages are the consumer's current private information and the firm's beliefs about that information; that is, $(M_1, S_1) = (\Theta, \Delta(\Theta))$ and $(M_2, S_2) = (\{v_L, v_H\}, \Delta(\{v_L, v_H\}))$, respectively. Second, the device in period t , φ_t , can be decomposed into two transition probabilities, β_t , which maps reports into distributions over beliefs, and α_t , which maps beliefs into distributions over allocations. For instance, in period 1, $\beta_1 : \Theta \mapsto \Delta(\Delta(\Theta))$ and $\alpha_1 : \Delta(\Theta) \mapsto \Delta(A_1)$.

Moreover, Theorem 2 in Doval and Skreta (2020) establishes that we can focus on PBE assessments such that we have the following (i) the consumer participates in the mechanism; (ii) conditional on participating, the consumer *truthfully* reports her type; (iii) if the mechanism outputs a belief F_2 , this is the belief that would result from Bayes' rule when the firm observes that message taking as given that the consumer participates and truthfully reveals her type.¹² Like in standard mechanism design, communication is direct since the consumer reveals her type to the mechanism. Furthermore, the mechanism encodes the information that the firm carries forward about the consumer's type in a Blackwell experiment.

¹²While optimal pricing in period 2 only depends on the likelihood of v_L over v_H , the revelation principle implies that the mechanism determines the belief that the firm has in period 1 about θ . This is similar to what happens in information design with evolving types, see Ely (2017).

The three constraints that the mechanism must satisfy, namely, the participation and truth-telling constraints for the agent and the Bayes' plausibility constraint, provide us with a tractable representation both of the consumer's behavior in a given period, as well as its impact on the mechanism offered in the next via the information that is generated about the consumer's type in the given period. Furthermore, the fact that, conditional on the induced belief, the allocation is determined independently of the consumer's type and depends only on the belief (the decomposition of φ_t into β_t and α_t), has a powerful implication: The allocation has to be measurable with respect to the information generated by the mechanism. Thus, the more the firm desires to tailor the allocation to the consumer's type, the more the firm has to learn about the consumer's type through the mechanism. As we illustrate next, this tension between the desire of the firm to offer personalized products and the information that is learned as a result is precisely what is at the heart of the optimal design of the product line.

4.1 Optimal mechanism under limited commitment

We now characterize the firm's optimal mechanism, starting from period 2. Following the *first order approach* that is prevalent in dynamic mechanism design and dynamic public finance (see Pavan et al., 2014; Stantcheva, 2020), we first solve a *relaxed* problem and then provide conditions under which the solution to the relaxed problem satisfies the remaining constraints.

4.1.1 Optimal mechanism in period 2

It is immediate to show that the optimal mechanism in period 2 coincides with the one we derived in [Section 3.1](#). That is, the firm chooses a price in period 2 as a function of the posterior mean of θ , μ_{F_2} .

In what follows, it is useful to keep track of whether it is optimal for the firm in period 2 to provide the good to a consumer with value v_L when the firm believes that θ is

distributed according to F_2 . We denote this by $q_2^*(v_L, F_2)$ and it satisfies the following:

$$q_2^*(v_L, F_2) = \begin{cases} 1 & \text{if } \mu_{F_2} < \bar{\mu} \\ 0 & \text{if } \bar{\mu} < \mu_{F_2} \end{cases}. \quad (1)$$

We specify $q_2^*(v_L, F_2)$ when $\mu_{F_2} = \bar{\mu}$ as part of the solution to the firm's problem in period 1, to which we turn next. While the firm is indifferent between setting a price of v_L or v_H when $\mu_{F_2} = \bar{\mu}$, from the perspective of period 1 one of the two prices may be optimal, so the indifference will be broken in favor of the period-1 firm.

4.1.2 Optimal mechanism in period 1

We now rely on the aforementioned observations to formulate a program whose solution yields the mechanism employed by the firm at the firm optimal PBE. Recall that a mechanism in period 1 is defined by two mappings:

$$\beta : \Theta \mapsto \Delta(\Delta(\Theta)), \alpha : \Delta(\Theta) \mapsto \Delta(A_1).$$

which assign to each type report, θ' , a distribution over posterior beliefs, $\beta(\cdot|\theta') \in \Delta(\Delta(\Theta))$, and to each posterior belief about θ , F_2 , a randomization over allocations, $\alpha(\cdot|F_2)$.

Instead of considering randomized allocations, we assume each posterior F_2 induces one quality level and one transfer, $(q_1(F_2), x_1(F_2))$. Because payoffs are quasilinear, considering mechanisms that do not randomize on the transfers is without loss of generality. While not allowing for randomization on the quality provision may be with loss of generality, it turns out to be without loss of optimality for the solution to the relaxed problem and hence, for the results that follow.

Given a mechanism, the consumer's payoff in period 1 when her type is θ and she truthfully reports θ is given by:

$$U(\theta) = \int_{\Delta(\Theta)} [\theta q_1(F_2) - x_1(F_2) + \theta \Delta v q_2^*(v_L, F_2)] \beta(dF_2|\theta), \quad (2)$$

where $q_2^*(v_L, F_2)$ is defined in Equation 1. To see how Equation 2 obtains, note that in period 2, the consumer makes a positive payoff only when her valuation is v_H and the firm sells the good at a price of v_L , in which case, she earns $v_H - v_L \equiv \Delta v$.

Theorem 2 in Doval and Skreta (2020) implies we can focus on mechanisms such that the following constraints hold. First, the consumer must report her type truthfully, so that

$$U(\theta) \geq \int_{\Delta(\Theta)} [\theta q_1(F_2) - x_1(F_2) + \theta \Delta v q_2^*(v_L, F_2)] \beta(dF_2|\tilde{\theta}), \quad (\text{L-IC}_{\theta, \tilde{\theta}})$$

where the right hand side of $(\text{L-IC}_{\theta, \tilde{\theta}})$ already incorporates that in period 2 a consumer whose private information is (θ, v) makes decisions optimally, even if she deviated in period 1 and reported $\tilde{\theta}$.

Second, the consumer must prefer to participate in the mechanism, so that for all $\theta \in \Theta$,

$$U(\theta) \geq 0. \quad (\text{L-PC}_{\theta})$$

To understand the right hand side of (L-PC_{θ}) , note the following. Because non-participation is an off-the-path event, Bayes' rule does not pin down the firm's beliefs about θ conditional on not participating. Thus, the firm can assign probability 1 to the consumer's type being $\theta = 1$ upon rejection. Thus, offering a price of v_H in period 2 is optimal. It follows that conditional on rejecting the mechanism in period 1, the consumer's payoff is 0.

Finally, the distribution over posteriors generated by the mechanism must be Bayes' plausible. That is, for all measurable subsets $\tilde{\Theta}$ of Θ and all measurable subsets \tilde{U} of $\Delta(\Theta)$, the following must hold:

$$\int_{\tilde{\Theta}} \beta(\tilde{U}|\theta) F_1(d\theta) = \int_{\tilde{\Theta}} \int_{\tilde{U}} F_2(\tilde{\Theta}) \beta(dF_2|\theta) F_1(d\theta). \quad (\text{BP}_{F_1})$$

Theorem 2 in Doval and Skreta (2020) implies the profit-maximizing mechanism can

be obtained by solving:

$$\max_{\beta, q_1, x_1} \int_{\Theta} \int_{\Delta(\Theta)} [x_1(F_2) - c(q_1(F_2)) + q_2^*(v_L, F_2)v_L + (1 - q_2^*(v_L, F_2))p(\theta)v_H] \beta(dF_2|\theta) F_1(d\theta), \quad (\text{L-OPT})$$

subject to the constraints, (L-IC $_{\theta, \hat{\theta}}$), (L-PC $_{\theta}$), and (BP $_{F_1}$). Note that the sequential rationality of the firm is fully captured in $q_2^*(v_L, F_2)$, which keeps track of how period-2 pricing depends on the information learned about the consumer in period 1.

In Section A.2, we show how to obtain an envelope representation of the consumer's utility $U(\theta)$, which we can use to replace the transfers out of the firm's profits. This representation, in turn, allows us to express the firm's expected profit in terms of virtual values and to reduce the mechanism design problem in period 1 to the problem of choosing two objects: a Bayes' plausible distribution over posteriors, $P_{\Delta(\Theta)}$, and for each posterior, a quality level, $q_1(F_2)$, to maximize

$$\int_{\Delta(\Theta)} \int_{\Theta} \left[\begin{array}{l} q_1(F_2) \left(\theta - \frac{(1-F_1(\theta))}{f_1(\theta)} \right) - c(q_1(F_2)) + (1 - q_2^*(v_L, F_2))p(\theta)v_H + \\ q_2^*(v_L, F_2) \left(p(\theta)v_H + (1 - p(\theta)) \left(v_L - \frac{p'(\theta)}{1-p(\theta)} \frac{1-F_1(\theta)}{f_1(\theta)} \Delta v \right) \right) \end{array} \right] F_2(d\theta) P_{\Delta(\Theta)}(dF_2), \quad (3)$$

subject to the constraints that

$$P_{\Delta(\Theta)} \text{ is Bayes plausible given } F_1, \quad (\text{BP})$$

$$U'(\theta) = \int_{\Delta(\Theta)} [q_1(F_2) + p'(\theta)\Delta v q_2^*(v_L, F_2)] \beta(dF_2|\theta) \text{ is increasing in } \theta. \quad (\text{MON})$$

It is easier to develop intuition about the dynamic virtual surplus without replacing the expressions for $p(\theta)$ and $F_1(\theta)$. The first two terms correspond to the static virtual surplus in the model of Mussa and Rosen (1978), but are expressed in terms of posterior beliefs. The second part of the virtual surplus corresponds to an adjusted version of the surplus in period 2. In period 2, with probability $1 - q_2^*(v_L, F_2)$, the firm trades only with the high-valuation consumer, in which case, a nonzero surplus exists only

when the consumer's valuation is v_H , which occurs with probability $p(\theta)$. With the remaining probability, the firm trades with the consumer regardless of her valuation, in which case, surplus is $p(\theta)v_H + (1 - p(\theta))v_L$. However, because whenever the firm trades with the low-valuation consumer, the high-valuation consumer makes rents, v_L is modified to reflect these rents. The adjustment by the rents is accounted for using the prior, and, importantly, it reflects the dynamic nature of the consumer's information. From the perspective of period 1, the firm only leaves rents for the second-period transaction because of the impact that the consumer's current type has on her future valuation. For this reason, the inverse hazard rate is multiplied by $p'(\theta)/(1 - p(\theta))$.

Remark 1. *Comparing the virtual surplus representation under limited commitment in Equation 3 with the one we derive in the case of commitment (see Equation A.4 in Section A.1) provides another way of understanding the difference between the problems. In both cases, the firm evaluates the optimality of a given mechanism using the "static" virtual values for θ and the "dynamic" virtual values for the period-2 valuations. The difference lies in that under commitment the firm chooses in period 1 the allocations for both periods. Instead, under limited commitment, the firm chooses the product line in period 1 taking into account that how much information is revealed through the product line affects its choice of price in period 2. This is captured by the term $q_2^*(v_L, F_2)$.*

Instead of fully solving the maximization problem implied by Equation 3, we focus for the rest of this section on the *relaxed problem*, where we drop the monotonicity constraint (MON). In addition to describing the solution to the relaxed problem in Propositions 2 and 3 below, we show that under a wide range of parameter configurations, the solution to the relaxed problem satisfies the monotonicity constraint (see Corollaries 1 and 2).

In the relaxed problem, for a given distribution over posteriors, $P_{\Delta(\Theta)}$, the firm chooses $q_1(F_2) = (2\mu_{F_2} - 1)/c$ when $\mu_{F_2} \geq 1/2$ and $q_1(F_2) = 0$, otherwise.¹³ Replacing the expressions for $c(\cdot)$, $p(\cdot)$, $F_1(\cdot)$, and the optimal choice of $q_1(F_2)$ reveals the firm's

¹³This is one of the steps where being able to separate the mechanism into a distribution over posteriors for each type, θ , and an allocation for each posterior belief, F_2 , is useful.

payoff is only a function of μ_{F_2} :

$$\int_{\Delta(\Theta)} \frac{(\max\{2\mu_{F_2} - 1, 0\})^2}{2c} P_{\Delta(\Theta)}(dF_2) + \int_{\Delta(\Theta)} [(1 - q_2^*(v_L, F_2))v_H\mu_{F_2} + q_2^*(v_L, F_2)(v_H\mu_{F_2} + (v_L - \Delta v)(1 - \mu_{F_2}))] P_{\Delta(\Theta)}(dF_2). \quad (4)$$

Since (4) depends only on μ_{F_2} , we can solve this problem with the tools in information design for continuum type spaces, which deal exclusively with the case in which the receiver's action and the sender's payoff are a function of the posterior mean (e.g., Gentzkow and Kamenica, 2016; Kolotilin, 2018; Dworzak and Martini, 2019).

To understand the trade-off introduced by the firm's limited commitment, it is instructive to first consider an artificial problem in which the firm can separately solve the problems represented in each line of Equation 4. That is, suppose the firm could choose $P_{\Delta(\Theta)}$ to maximize the first line and $P'_{\Delta(\Theta)}$ to maximize the second line.

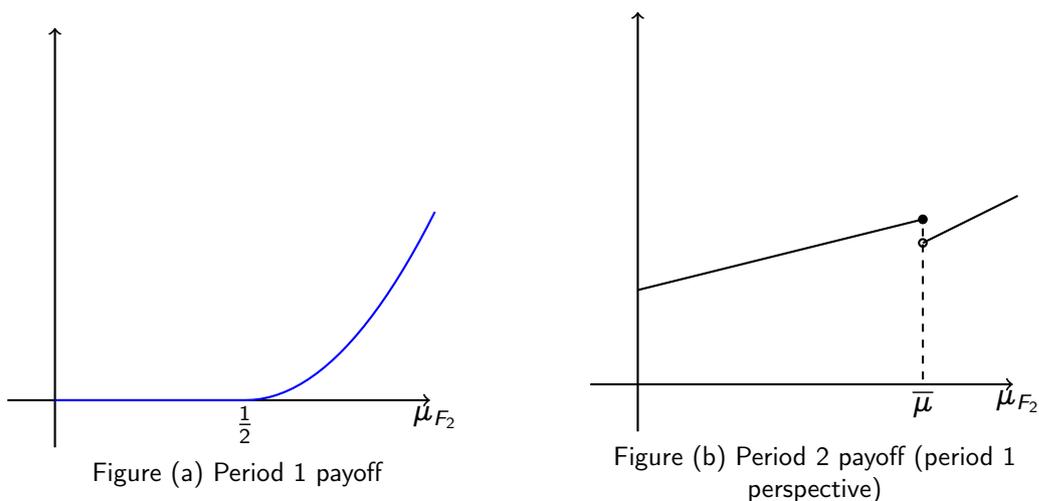


Figure 6: The firm's payoff as a function of the posterior mean
 $((v_H, v_L, c) = (1, 0.75, 2))$

Figure 6 illustrates the firm's payoff as a function of the posterior mean, μ_{F_2} , in each of these problems when $\bar{\mu} > 1/2$. Figure 6a plots the integrand in the first line of Equation 4. Figure 6b plots the integrand in the second line of Equation 4. In period 2, the firm sets a price of v_H when $\mu_{F_2} > \bar{\mu}$, and v_L otherwise. In period 1, because

$\bar{\mu} > 1/2$, the firm prefers the price in period 2 to be v_L at $\mu_{F_2} = \bar{\mu}$, so we break ties accordingly.

For the purposes of maximizing its profit in period 1, the firm prefers a distribution over posterior means that perfectly reveals the types above $1/2$. This preference is intuitive: The period-1 problem coincides with the linear-quadratic version of [Mussa and Rosen \(1978\)](#), which features full separation. Perfectly tailoring the quality provided to the consumer's type is optimal for the firm.

By contrast, when $\bar{\mu} < 1/2$, the firm in period 1 prefers that the price in period 2 to be v_L *regardless* of the induced posterior mean. This can be achieved by not disclosing any information about θ . Because the prior mean of θ is less than $\bar{\mu}$, by not disclosing any information about θ , the firm in period 1 can guarantee that the period 2 price is v_L with probability 1.

When the firm has commitment, it can obtain its maximum value in both problems: It can implement the Mussa-Rosen solution in period 1, and, ignoring the information about θ revealed by the allocation, it can set a period 2 price equal to v_L . Indeed, the mechanism described in the previous two paragraphs is precisely the commitment solution presented in [Section 3.2](#). If using the information revealed by the allocation is detrimental to revenue in period 2, the firm can commit to ignore it.

However, when the firm has limited commitment, the allocation has to be measurable with respect to the information released by the mechanism. Thus, to achieve the maximum payoff in period 1, the firm must bear the cost of pricing all types above $1/2$ at a price of v_H . Instead, to achieve the maximum payoff in period 2, the firm needs to reveal no information in period 1, which in this example implies all consumer types receive the lowest quality good in period 1.

Not surprisingly, the optimal mechanism turns out to be a compromise between these two forces. Except when $\bar{\mu} \leq 1/4$, in which the firm can obtain the commitment payoff, the optimal mechanism distorts quality provision in period 1 to discipline the revelation of information about θ across periods. To describe the optimal mechanism, we consider separately the cases $\bar{\mu} < 1/2$ and $\bar{\mu} > 1/2$.

Case 1: $\bar{\mu} < 1/2$ In the commitment solution, when $\bar{\mu} < 1/2$, the firm prefers to set a price equal to v_H in period 2. Under limited commitment, the firm only sets a price of v_H in period 2 if the posterior mean of θ is above $\bar{\mu}$. Thus, to maintain high prices in period 2, the consumer's purchase decision in period 1 must convince the firm that the consumer's type is sufficiently high.

Figure 7 illustrates how the firm alters its product line to maintain high prices in period 2:

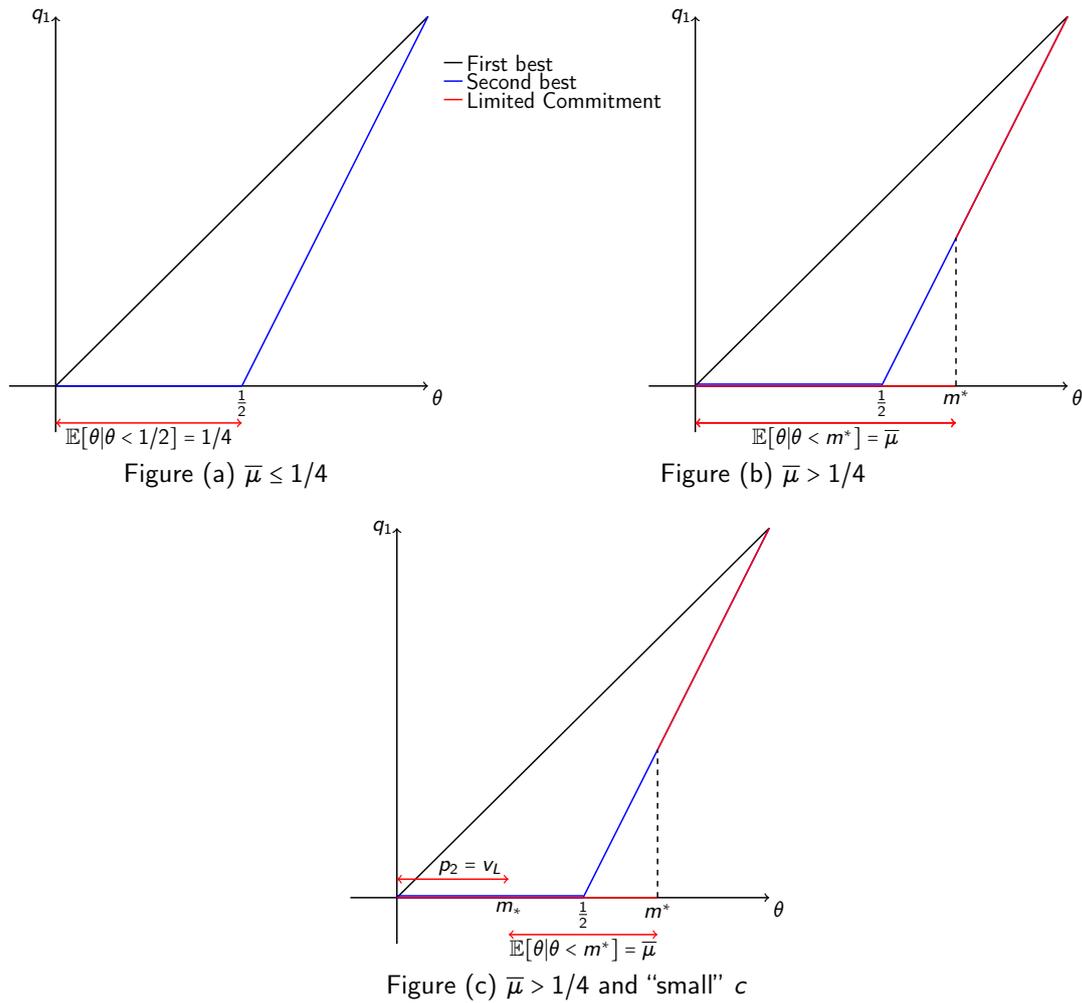


Figure 7: Product choice when $\bar{\mu} < 1/2$ in first best (black), commitment (blue), and limited commitment (red)

When $\bar{\mu} \leq 1/4$, the firm can implement the optimal mechanism with commitment even under limited commitment. To see this, suppose that the firm only observed the quality purchased by the consumer under the commitment solution. For consumers with $\theta \geq 1/2$, the firm would perfectly learn θ and since $\theta \geq 1/2 > \bar{\mu}$, it is optimal to set a price of v_H in period 2. For consumers with $\theta < 1/2$, the firm would only learn that they purchased the good of the lowest quality, so on average θ is a $1/4$. Since $\bar{\mu} \leq 1/4$, it is also optimal for the firm to set a price of v_H in period 2. Thus, if the mechanism only reveals the quality purchased in period 1, the firm can then maintain high prices in period 2. Note that whereas the firm can implement the commitment solution, limited commitment shapes how information is transmitted across periods. Indeed, when $\bar{\mu} \leq 1/4$ there is a sense in which the commitment solution reveals “too much” information: consumer types below $1/2$ reveal θ to the mechanism which is used neither for product customization in period 1 nor for price discrimination in period 2.

Instead, when $\bar{\mu} > 1/4$, it is not enough that consumer types below $1/2$ purchase the lowest quality good for the firm to maintain high prices in period 2. Thus, the firm chooses to remove some fairly low quality products so that both low and middle consumer types purchase the good of the lowest quality in period 1 (i.e., $q_1 = 0$). That is, relative to the optimal mechanism under commitment, the firm *prunes* its product line under limited commitment. In this way, the decision to purchase the lowest quality good no longer reveals that the consumer’s type is low, therefore allowing the firm to maintain high prices in period 2. Indeed, as illustrated in [Figure 7b](#), the firm pools all consumer types that buy the good of the lowest quality in period 1 and offers all consumers a price of v_H in period 2 regardless of their period-1 purchase.

When $\bar{\mu} \in (1/4, 1/2)$ and the firm’s mechanism is as illustrated in [Figure 7b](#), both the firm and the consumer lose from the firm’s limited commitment. Note the consumer faces the same price in period 2 as in the commitment solution. However, in period 1, a consumer with type below m^* in [Figure 7b](#) receives the lowest quality good. Furthermore, a consumer with type above m^* faces higher prices in period 1. Indeed, by pruning products from the product line, the firm gives the consumer fewer opportunities to self-select in period 1. Therefore, the firm needs to leave less rents to the consumer

in period 1, and hence charges higher prices. Despite the possibility of charging higher prices, the firm is clearly worse off because it cannot implement the commitment solution.

However, the closer $\bar{\mu}$ is to $1/2$ the more the firm needs to prune the product line to prevent price discrimination in period 2. Thus, when $\bar{\mu}$ is high or the cost of product personalization is low (see Equation 5), the firm would prefer to learn more than just whether the consumer purchased a good of the lowest quality in period 1. Figure 7c illustrates the solution to the relaxed problem in this case: The firm separates the lowest consumer types that buy $q_1 = 0$ (i.e., those in $[0, m_*)$ from the low-to-middle consumer types that buy $q_1 = 0$, (i.e., those in $[m_*, m^*]$), offering the former a price of v_L in period 2. In turn, this allows the firm not to sacrifice product personalization for the high consumer types in period 1 (i.e., those above m^*), since the firm no longer needs to pool them with the lowest types to keep a price of v_H in period 2. It is immediate to see that it is not possible to find transfers to implement the allocation that solves the relaxed problem: Since the types below m_* receive a price of v_L in period 2, higher consumer types must be paid rents upfront for the forgone rents in period 2. In turn, types below m_* would prefer to announce that they have types in $[m_*, m^*]$ to seize these upfront rents:

Proposition 2 summarizes the solution to the relaxed problem when $\bar{\mu} < 1/2$ and Corollary 1 provides conditions under which the solution to the relaxed problem can be achieved at the firm optimal PBE:

Proposition 2. *Assume $\bar{\mu} < 1/2$. Then, the following is the solution to the relaxed problem:*

1. *If $\bar{\mu} \leq 1/4$, consumer types above $1/2$ purchase a good with quality $\frac{2\theta-1}{c}$, while consumer types below $1/2$ purchase a good with quality 0. In period 2, the price is v_H independently of the quality purchased in period 1.*

2. Instead, if $\bar{\mu} > 1/4$ and

$$2c \geq \frac{(4\bar{\mu} - 1)^2}{1 - 2\bar{\mu}}, \quad (5)$$

let $m^* \geq 1/2$ be the unique threshold such that

$$\bar{\mu} = \mathbb{E}[\theta | \theta \leq m^*].$$

Then, consumer types above m^* purchase a good with quality $\frac{2\theta-1}{c}$, while consumer types below m^* purchase a good with quality 0. In period 2, the price is v_H independently of the quality purchased in period 1.

3. Finally, if $\bar{\mu} \in (1/4, 1/2)$ and Equation 5 does not hold, let m_*, m^* be such that¹⁴

$$\mathbb{E}[\theta | \theta \in [m_*, m^*]] = \bar{\mu}.$$

Then, consumer types above m^* purchase a good with quality $\frac{2\theta-1}{c}$, while consumer types below m^* purchase a good with quality 0. In period 2, the price is v_H for $\theta \geq m_*$ and v_L otherwise.

In cases 1 and 2, the firm's information about θ is summarized by the consumer's product choice in period 1. That is, if the consumer receives quality q , then the firm believes that the consumer's valuation is v_H with probability $\mathbb{E}[\theta | \{\theta \in [0, 1] : \theta \in \text{supp } F_2 \text{ and } q_1(F_2) = q\}]$. Instead, in case 3, conditional on the consumer purchasing $q_1 = 0$, the firm also learns whether θ is below m_* or above m_* .

Corollary 1. Suppose $\bar{\mu} \in [0, 1/2)$. Then, the solution to the relaxed problem satisfies the monotonicity constraints in cases 1 and 2 in Proposition 2. Therefore, finding transfers in period 1 such that the solution to the relaxed problem can be achieved at the firm optimal PBE is possible.

As previously discussed, in case 3, the firm would like to solve the trade-off between personalization and preventing price discrimination in favor of personalization. Antici-

¹⁴ m_*, m^* are uniquely pinned down by Equation A.17 in Section A.2.

pating this, the consumer demands rents upfront that prevent the firm from being able to implement the solution to the relaxed problem.

Case 2: $\bar{\mu} \geq 1/2$ In the commitment solution, when $\bar{\mu} \geq 1/2$, the firm prefers to set a price equal to v_L in period 2. Under limited commitment, the firm only sets a price of v_L in period 2 if the posterior mean of θ is below $\bar{\mu}$. Thus, to maintain low prices in period 2, the consumer's purchase decision in period 1 must convince the firm that the consumer's type is sufficiently low. Figure 8 illustrates how the firm alters its product line to maintain low prices in period 2. Interestingly, in this case, the firm price discriminates in period 2: Consumers who purchase products on the high end of the product line pay high prices in period 2.

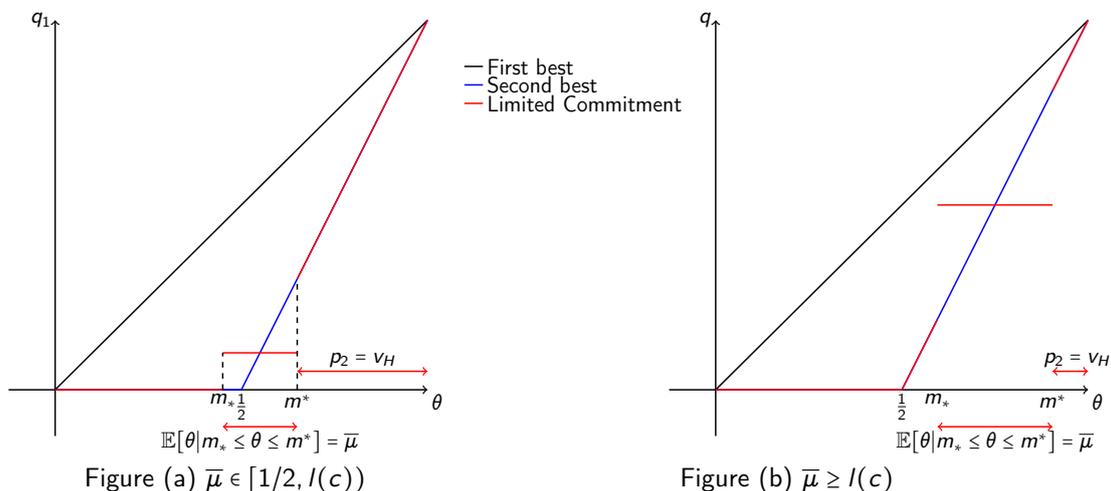


Figure 8: Product choice when $\bar{\mu} \geq 1/2$ in first best (black), commitment (blue), and limited commitment (red)

When the firm wishes to sustain low prices in period 2, relative to the optimal mechanism with commitment, it prunes qualities in the middle of the product line. As Figure 8 illustrates, this pruning forces middle-to-high consumer types to purchase lower qualities than in the commitment solution, whereas some low-to-middle consumer types purchase a higher quality than they would have in the commitment solution. Indeed, as illustrated in Figure 8a, consumer types who buy the lowest-quality good under the commitment solution may buy goods of higher quality under limited commitment. By

pooling mid-low and mid-high consumer types in period 1, the firm is able to offer those types a low price in period 2.

Note the firm does not distort the high end of the product line, so that when a consumer purchases the highest quality products, they face a period 2 price of v_H . For these consumers, the cost of price discrimination is lower than the cost of not providing them with a product tailored to their tastes (recall [Figure 6a](#)).

Because high consumer types anticipate they will receive no rents in period 2, incentive compatibility in period 1 implies the firm must compensate them for the rents, $\theta\Delta v \propto \theta(1-\bar{\mu})$, that they would obtain by pretending to be a lower type consumer. When $\bar{\mu}$ is small, these upfront rents are “tempting” for low consumer types in period 1, who may now wish to report that they value quality in period 1 more than they actually do.¹⁵ In other words, the solution to the relaxed problem may fail to satisfy the monotonicity constraint for low values of $\bar{\mu}$. [Proposition 3](#) summarizes the solution to the relaxed problem when $\bar{\mu} > 1/2$ and [Corollary 2](#) provides conditions under which the solution to the relaxed problem can be achieved at the firm optimal PBE:

Proposition 3. *Assume $\bar{\mu} \geq 1/2$. Furthermore, let m_* , m^* be such that¹⁶*

$$\mathbb{E}[\theta | \theta \in [m_*, m^*]] = \bar{\mu},$$

and let $I(c) = (2 + 3c)/4(1 + c)$. The solution to the relaxed program is as follows:

1. If $1/2 \leq \bar{\mu} < I(c)$, consumer types below m_* purchase the lowest quality good in period 1 and receive a period-2 price of v_L . Consumer types $\theta \in [m_*, m^*]$ receive quality $(2\bar{\mu} - 1)/c$ and a period-2 price of v_L . Consumer types above m^* receive quality $(2\theta - 1)/c$ and receive a period-2 price of v_H .
2. If $I(c) \leq \bar{\mu} < 1$, consumer types below $1/2$ purchase the lowest quality good, consumer types in $[1/2, m_*) \cup (m^*, 1]$ receive quality $(2\theta - 1)/c$, and consumer

¹⁵The above logic is reminiscent of the “take the money and run” strategy in Laffont and Tirole (1988).

¹⁶When $\bar{\mu} = 1/2$, $m_* = m^* = 1/2$. Otherwise, $m_* < m^*$ and as we show in [Section A.2](#), they are uniquely pinned down as a function of the parameters, (v_L, v_H, c) .

types in $[m_*, m^*]$ receive quality $(2\bar{\mu} - 1)/c$. In period 2, consumer types below m^* receive a price of v_L , and otherwise receive a price of v_H .¹⁷

In both cases, the firm's information about θ is summarized by the consumer's product choice in period 1. That is, if the consumer receives quality q , then the firm believes that the consumer's valuation is v_H with probability $\mathbb{E}[\theta | \{\theta \in [0, 1] : \theta \in \text{supp } F_2 \text{ and } q_1(F_2) = q\}]$.

Corollary 2. *Suppose that $\bar{\mu} \geq 0.5 + c/4$. Then, the solution to the relaxed problem satisfies the monotonicity constraint, (MON). Therefore, finding transfers in period 1 such that the solution to the relaxed problem can be achieved at the firm optimal PBE is possible.*

Implementation: The results in Propositions 2 and 3 shows that the firm introduces distortions in its product line relative to the commitment solution in an attempt to obfuscate how much it learns about the consumer's preference for quality in period 1. Indeed, as described above, the optimal product line sorts the different consumer types in (sometimes multiple) separation and pooling intervals. Despite this, whenever the monotonicity constraints hold, the firm's optimal mechanism has a *simple* implementation. Indeed, the firm can offer the consumer a menu of qualities and payments in period 1, such that what the firm learns from observing the consumer's choices from the menu coincides with the information that is induced by the optimal mechanism.

A major challenge in the extant literature on limited commitment is how to keep track of how the consumer's best response to the mechanism affects the information that the firm obtains from the interaction, which in turn affects the firm's incentives to offer the mechanism in the first place. Instead, leveraging the results in Doval and Skreta (2020) allows us to reduce the consumer's best response to the firm's mechanism and its informational feedback to a set of constraints that the mechanism must satisfy. Thus, instead of having to consider complicated mixed strategies for the consumer in response to the firm's offer of a mechanism (as in, e.g., Laffont and Tirole, 1988),

¹⁷When $\bar{\mu} = 1$, we have that $v_L = v_H$ and there is no price discrimination in period 2. Thus, the limited commitment solution coincides with the commitment solution.

the feedback between the consumer's behavior and the firm's optimal period-2 pricing is transformed into an information design problem. Leveraging techniques from that literature we can obtain a solution that involves a distribution of posterior means that we then use to back out the optimal menu.

5 Concluding remarks

We study how the ability to engage in price discrimination shapes the design of the product line. In doing so, we uncover a novel channel that is present in settings where a firm interacts repeatedly with its customers. A rich array of products allows for higher customer satisfaction, but, the mere choice of the product carries valuable information about the consumer that the firm can leverage for price discrimination. As we mentioned in [Section 1](#), this trade-off between the benefits and the costs of product personalization is at the center of the policy debate regarding the use of personalized data.¹⁸ Previous studies have focused on the information revealed by accepting to pay a price to purchase a fixed product variety. Our work shows that the ratcheting frictions identified in those previous studies also play an important role in *shaping* the design of a firm's product line.

Our work opens up several avenues for future research. First, the strength of the ratchet effect, and hence the extent to which the firm may prune its product line, depends on how forward looking consumers are. Considering the effects of having consumers of different levels of sophistication in the firm's optimal mechanism would be interesting. Second, our results imply that in terms of the product line, our model is observationally distinguishable from that of [Mussa and Rosen \(1978\)](#). This implication puts at the forefront a concern in empirical work about the endogeneity of the set of options consumers face (e.g., [Ivaldi and Martimort, 1994](#); [Miravete, 2002](#); [Luo et al., 2018](#)), making salient the possibility that this endogeneity may be pervasive in settings where firms and consumers interact repeatedly over time.

Finally, while our analysis relies on a number of parametric assumptions whose only role

¹⁸See Executive Office of the President of the United States (2015).

is to allow us to apply the existing methodology of information design for continuum type spaces, the economic force underlying the optimal product line is likely to extend to more general settings. Indeed, an interpretation of our model is that, faced with the dynamic inconsistency of the commitment solution, the firm in period 1 prefers to acquire less information about the consumer as a self-disciplining device, as in Carrillo and Mariotti (2000). While it is natural to conjecture that this economic force extends to more general settings, showing this formally requires extending the existing toolkit of information design with continuum type spaces.¹⁹ Because it will enable a deeper exploration of the issues raised in this paper and open the analysis of new problems, we see this extension as a fruitful avenue for further research.

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¹⁹Onuchic and Ray (2021) is a step in this direction, where they characterize optimal monotone partitions in an information design setting with a continuum type space. However, the results in Proposition 2 and Proposition 3 highlight that in our setting mechanisms that induce monotone partitions may not be incentive feasible.

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A Omitted proofs

Remark A.1. *Throughout, we make the following technical assumptions. Unless noted otherwise, all spaces are Polish spaces; we endow them with their Borel σ -algebra. Second, product spaces are endowed with their product σ -algebra. Third, for a Polish space X , we let $\Delta(X)$ denote the set of Borel probability measures over X , endowed with the weak* topology. Thus, $\Delta(X)$ is also Polish (Aliprantis and Border (2013)). Finally, for any two measurable spaces X and Y , a mapping $\varphi : X \mapsto \Delta(Y)$ is a transition probability from X to Y if, for any measurable $C \subseteq Y$, $\varphi(C|x) \equiv \varphi(x)(C)$ is a measurable real valued function of $x \in X$.*

A.1 Proofs of Section 3.2

One can use (F-IC $_{\theta',v,v'}$) to obtain a monotonicity condition,

$$v_L(q_2(\theta, v_H) - q_2(\theta, v_L)) \leq x_2(\theta, v_H) - x_2(\theta, v_L) \leq v_H(q_2(\theta, v_H) - q_2(\theta, v_L)), \quad (\text{A.1})$$

that we use later on. Furthermore, since $p(\cdot)$ is differentiable, we can apply the envelope theorem in Milgrom and Segal (2002) to obtain the following envelope condition from (F-IC $_{\theta,\theta'}$):

$$\frac{\partial}{\partial \theta} U_1(\theta', \theta)|_{\theta'=\theta} = q_1(\theta) + p'(\theta)\Delta(\theta), \quad (\text{A.2})$$

where $\Delta(\theta) \equiv v_H q_2(\theta, v_H) - v_L q_2(\theta, v_L) + x_2(\theta, v_L) - x_2(\theta, v_H)$.

The firm's profits are given by

$$\int_{\Theta} [x_1(\theta) - c(q_1(\theta)) + p(\theta)x_2(\theta, v_H) + (1 - p(\theta))x_2(\theta, v_L)] F_1(d\theta). \quad (\text{A.3})$$

Using the envelope representation of payoffs, we obtain the following version of the dynamic virtual surplus:

$$\int_{\Theta} \left[\begin{aligned} & q_1(\theta) \left(\theta - \frac{(1-F_1(\theta))}{f_1(\theta)} \right) - c(q_1(\theta)) + v_H q_2(\theta, v_H) \left(p(\theta) - p'(\theta) \frac{1-F_1(\theta)}{f_1(\theta)} \right) + \\ & v_L q_2(\theta, v_L) \left((1-p(\theta)) + \frac{1-F_1(\theta)}{f_1(\theta)} p'(\theta) \right) q_2(\theta, v_L) \\ & + \frac{1-F_1(\theta)}{f_1(\theta)} p'(\theta) (x_2(\theta, v_H) - x_2(\theta, v_L)) \end{aligned} \right] F_1(d\theta). \quad (\text{A.4})$$

In the above expression, we have already replaced that in a profit-maximizing mechanism $U(0) = 0$.

In the setting we are considering, $p(\theta) = \theta$ —higher types are more likely to like the product more tomorrow—and $p' > 0$, so we want to set the difference in transfers as big as possible; that is, we set

$$x_2(\theta, v_H) - x_2(\theta, v_L) = v_H(q_2(\theta, v_H) - q_2(\theta, v_L)),$$

which implies $\Delta(\theta) = q_2(\theta, v_L)\Delta v$. Replacing the upper bound in [Equation A.1](#), we obtain

$$\int_{\Theta} \left[q_1(\theta) \left(\theta - \frac{(1-F_1(\theta))}{f_1(\theta)} \right) - c(q_1(\theta)) + p(\theta)v_H q_2(\theta, v_H) + (1-p(\theta))q_2(\theta, v_L) \left(v_L - \Delta v \frac{1-F_1(\theta)}{f_1(\theta)} \frac{p'(\theta)}{1-p(\theta)} \right) \right] F_1(d\theta). \quad (\text{A.5})$$

Replacing the parametric assumptions, this leads to the following expression for the dynamic virtual surplus:

$$\int_{\Theta} \left[q_1(\theta)(2\theta - 1) - \frac{c}{2}(q_1(\theta))^2 + \theta v_H q_2(\theta, v_H) + (1-\theta)q_2(\theta, v_L)(2v_L - v_H) \right] F_1(d\theta).$$

The above expression is maximized by setting the following:

1. $q_1(\theta) = \frac{2\theta-1}{c}$ whenever $\theta \geq 1/2$, and 0 otherwise,
2. $q_2(\theta, v_H) = 1$,
3. $q_2(\theta, v_L) = 1$ if $\bar{\mu} > 1/2$, and 0 otherwise.

It is a standard to retrieve the transfers from the consumer's envelope representation of payoffs and check that the incentive compatibility constraint is satisfied.

A.2 Proofs of Section 4

Envelope theorem: We first argue that

$$U(\theta) = \int_{\Delta(\Theta)} [\theta q_1(F_2) - x_1(F_2) + \theta \Delta v q_2^*(v_L, F_2)] \beta(dF_2|\theta),$$

is Lipschitz continuous, and hence almost everywhere differentiable. To see this, consider the payoff from the following deviation: The consumer with type θ reports $\tilde{\theta}$ and then follows the strategy of $\tilde{\theta}$ in period 2. Her payoff would then be given by $U(\tilde{\theta}, \theta)$ as defined in the main text.

The optimality of truthtelling implies

$$U(\theta) = \max_{\tilde{\theta} \in \Theta} U(\tilde{\theta}, \theta).$$

We now establish that the family $\{U(\tilde{\theta}, \cdot) : \tilde{\theta} \in \Theta\}$ is equi-Lipschitz continuous. Let θ and θ' be such that $\theta \neq \theta'$, and consider

$$\begin{aligned} |U(\tilde{\theta}, \theta) - U(\tilde{\theta}, \theta')| &= \left| \int_{\Delta(\Theta)} ((\theta - \theta')q_1(F_2) + (\theta - \theta')\Delta v q_2^*(v_L, F_2)) \beta(dF_2|\tilde{\theta}) \right| \\ &\leq |\theta - \theta'| \int_{\Delta(\Theta)} [q_1(F_2) + \Delta v q_2^*(v_L, F_2)] \beta(dF_2|\tilde{\theta}) \leq |\theta - \theta'| (\max_{q_1} q_1 + \Delta v). \end{aligned}$$

Because $\max q_1 = \bar{Q}$ is bounded by assumption, the result follows. Then, U is Lipschitz continuous, because it is the max over a family of equi-Lipschitz continuous functions. Moreover, at any point of differentiability of $U(\cdot)$, we have

$$U'(\theta) = \int_{\Delta(\Theta)} [q_1(F_2) + p'(\theta)\Delta v q_2^*(v_L, F_2)] \beta(dF_2|\theta). \quad (\text{A.6})$$

Incentive compatibility implies $U'(\theta)$ is nondecreasing. Equation A.6 implies

$$\begin{aligned} \int_{\Theta} \int_{\Delta(\Theta)} x_1(F_2) \beta(dF_2|\theta) F_1(d\theta) &= \int_{\Theta} \int_{\Delta(\Theta)} [\theta q_1(F_2) + p(\theta) \Delta v q_2^*(v_L, F_2)] \beta(dF_2|\theta) F_1(d\theta) \\ &- \int_{\Theta} \int_0^{\theta} \left(\int_{\Delta(\Theta)} [q_1(F_2) + p'(\theta) \Delta v q_2^*(v_L, F_2)] \beta(dF_2|u) \right) du F_1(d\theta). \end{aligned} \quad (\text{A.7})$$

Virtual surplus: The firm's profits are given by:

$$\int_{\Theta} \int_{\Delta(\Theta)} [x_1(F_2) - c(q_1(F_2)) + (1 - q_2^*(v_L, F_2)) p(\theta) v_H + q_2^*(v_L, F_2) v_L] \beta(dF_2|\theta) F_1(d\theta).$$

Replacing Equation A.7 and integrating by parts, we obtain

$$\int_{\Theta} \int_{\Delta(\Theta)} \left[\begin{aligned} &q_1(F_2) \left(\theta - \frac{(1-F_1(\theta))}{f_1(\theta)} \right) - c(q_1(F_2)) + (1 - q_2^*(v_L, F_2)) p(\theta) v_H \\ &+ q_2^*(v_L, F_2) \left(p(\theta) v_H + (v_L - \Delta v \frac{1-F_1(\theta)}{f_1(\theta)} \frac{p'(\theta)}{1-p(\theta)}) \right) \end{aligned} \right] \beta(dF_2|\theta) F_1(d\theta). \quad (\text{A.8})$$

Denote by P the distribution on $\Theta \times \Delta(\Theta)$ defined as $P(\tilde{\Theta} \times \tilde{U}) = \int_{\tilde{\Theta}} \beta(\tilde{U}|\theta) F_1(d\theta)$, for all measurable subsets $\tilde{\Theta}, \tilde{U}$ of Θ and $\Delta(\Theta)$. Letting $P_{\Delta(\Theta)}$ denote its marginal on $\Delta(\Theta)$, Proposition 3.6 in Crauel (2002) implies Equation A.8 equals²⁰

$$\int_{\Delta(\Theta)} \int_{\Theta} \left[\begin{aligned} &q_1(F_2) \left(\theta - \frac{(1-F_1(\theta))}{f_1(\theta)} \right) - c(q_1(F_2)) + (1 - q_2^*(v_L, F_2)) p(\theta) v_H \\ &+ q_2^*(v_L, F_2) \left(p(\theta) v_H + (1 - p(\theta)) \left(v_L - \frac{p'(\theta)}{1-p(\theta)} \frac{1-F_1(\theta)}{f_1(\theta)} \Delta v \right) \right) \end{aligned} \right] F_2(d\theta) P_{\Delta(\Theta)}(dF_2), \quad (\text{A.9})$$

which is the expression in Equation 3. The optimal mechanism for the firm is the one that maximizes Equation A.9 subject to $P_{\Delta(\Theta)}$ being Bayes plausible and $U'(\theta)$ being nondecreasing.

Relaxed problem: Ignoring the monotonicity condition, we can choose $q_1(F_2)$ to maximize pointwise the integrand in Equation A.9, in which case, we obtain

$$q_1(F_2) = \max \left\{ 0, \frac{2\mu_{F_2} - 1}{c} \right\} \equiv q_1(\mathbb{E}_{F_2}(\theta)). \quad (\text{A.10})$$

²⁰Recall the assumptions in Remark A.1.

Replacing Equation A.10 in Equation A.9 leads to Equation 4, which we reproduce below for ease of reference:

$$\int_{\Delta(\Theta)} \frac{(\max\{2\mu_{F_2} - 1, 0\})^2}{2c} P_{\Delta(\Theta)}(dF_2) + \int_{\Delta(\Theta)} [(1 - q_2^*(v_L, F_2))v_H\mu_{F_2} + q_2^*(v_L, F_2)(v_H\mu_{F_2} + (v_L - \Delta v)(1 - \mu_{F_2}))] P_{\Delta(\Theta)}(dF_2). \quad (\text{A.11})$$

It follows that we can write the virtual surplus as a function of the posterior mean of F_2 , and thus we can pool together all distributions F_2 which induce the same posterior mean. Thus, letting m denote the posterior mean, the firm's payoff can be written as:

$$\int_0^1 R(m)G(dm),$$

where G is a distribution on $[0, 1]$ that is dominated by F_1 in the convex order, and the function R is defined as follows. If $\bar{\mu} < 1/2$, we have

$$R(m) = \begin{cases} 2m\Delta v + 2v_L - v_H & \text{if } 0 \leq m < \bar{\mu} \\ mv_H & \text{if } \bar{\mu} \leq m \leq 1/2 \\ mv_H + \frac{(2m-1)^2}{2c} & \text{if } 1/2 \leq m \end{cases},$$

and if $\bar{\mu} \geq 1/2$, we have

$$R(m) = \begin{cases} 2m\Delta v + 2v_L - v_H & \text{if } 0 \leq m \leq \frac{1}{2} \\ 2m\Delta v + 2v_L - v_H + \frac{(2m-1)^2}{2c} & \text{if } \frac{1}{2} < m \leq \bar{\mu} \\ mv_H + \frac{(2m-1)^2}{2c} & \text{otherwise} \end{cases}.$$

The solution to the relaxed problem can be obtained from solving

$$\max_{G: F_1 \succ_{cx} G} \int_0^1 R(m)G(dm). \quad (\text{A.12})$$

Because R satisfies the conditions of Dworczak and Martini (2019), it follows from their work that G^* is a solution to Equation A.12 if and only if a convex function π exists such that (i) $\mathbb{E}_{G^*}\pi(m) = \mathbb{E}_{F_1}\pi(m)$, (ii) $\pi \geq R$, and (iii) $\text{supp } G^* \subseteq \{m : \pi(m) = R(m)\}$. The proof of Propositions 2 and 3 proceeds as follows. In each case, we prove that the

induced distribution over posterior means is optimal by constructing a convex function π and verifying that the above conditions hold. We then use [Equation A.7](#) to construct the transfers and provide conditions under which the solution satisfies that $U'(\theta)$ is nondecreasing.

Proof of Proposition 2. When $\bar{\mu} \leq 1/4$, [Proposition 2](#) implies that the firm's optimal mechanism induces the following distribution over posterior means:

$$G(m) = \begin{cases} 0 & \text{if } m < 1/4 \\ F_1(1/2) & \text{if } 1/4 \leq m \leq 1/2 \\ F_1(m) & \text{otherwise} \end{cases} .$$

It is immediate to check that G is supported by the following convex function:

$$\pi(m) = \begin{cases} mv_H & \text{if } m \leq 1/2 \\ R(m) & \text{otherwise} \end{cases} .$$

Because $\mathbb{E}[\theta|\theta \leq 1/2] = 1/4 > \bar{\mu}$, the period 2 firm sets a price equal to v_H for all realizations of m . Consumer types with $\theta < 1/2$ are excluded in period 1, whereas they receive $\frac{2\theta-1}{c}$ and pay $\frac{\theta^2-1/4}{c}$ if $\theta \geq 1/2$. It is immediate to show that $U'(\theta)$ is monotone.

Suppose now that $1/4 < \bar{\mu} < 1/2$. In what follows, we normalize v_H to 1 to simplify notation. We consider two cases depending on whether the following inequality holds:

$$1 - 2\bar{\mu} \geq \frac{(4\bar{\mu} - 1)^2}{2c} \tag{A.13}$$

If [Equation A.13](#) holds, then the optimal disclosure policy reveals whether θ is above or below m^* , where m^* solves $\mathbb{E}[\theta|\theta \leq m^*] = \bar{\mu}$, that is $m^* = 2\bar{\mu}$. The induced distribution over posteriors G and the convex function that supports it are given by:

$$G(m) = \begin{cases} 0 & \text{if } m < \bar{\mu} \\ F_1(m^*) & \text{if } \bar{\mu} \leq m \leq m^* \\ F_1(m) & \text{otherwise} \end{cases} , \quad \pi(m) = \begin{cases} R(\bar{\mu}) + \frac{R(m^*)-R(\bar{\mu})}{m^*-\bar{\mu}}(m-\bar{\mu}) & \text{if } m \leq m^* \\ R(m) & \text{otherwise} \end{cases} . \tag{A.14}$$

The inequality in Equation A.13 ensures that $\pi(m) \geq R(m)$ for all $m \in [0, 1]$.

The disclosure policy described in Equation A.13 implies that in the solution to the relaxed problem types below m^* are excluded in period 1, whereas types above m^* receive $(2\theta - 1)/c$ and pay $\frac{\theta^2 - m^*(1 - m^*)}{c}$. It is immediate to show $U'(\theta)$ is monotone.

Suppose now that Equation A.13 does not hold. Then, the optimal disclosure policy reveals whether θ is in $[m_*, m^*]$ and in case $\theta \notin [m_*, m^*]$, it fully reveals θ . The thresholds $0 \leq m_* < \bar{\mu} \leq 1/2 \leq m^*$ satisfy that

$$\mathbb{E}[\theta | \theta \in [m_*, m^*]] = \bar{\mu},$$

and the implied distribution over posteriors satisfies:

$$G(m) = \begin{cases} F_1(m) & \text{if } 0 \leq m < m_* \text{ or } m^* \leq m \leq 1 \\ F_1(m_*) & \text{if } x \in [m_*, \bar{\mu}) \\ F_1(m^*) & \text{if } x \in [\bar{\mu}, m^*) \end{cases}. \quad (\text{A.15})$$

Assume first that m_*, m^* exist. We construct a convex function $\pi(m)$ that satisfies the necessary properties. Define $\pi : [0, 1] \mapsto \mathbb{R}$ as follows:

$$\pi(m) = \begin{cases} R(m) & \text{if } m \notin [m_*, m^*] \\ R(m_*) + \frac{R(m^*) - R(m_*)}{m^* - m_*}(m - m_*) & \text{otherwise} \end{cases}. \quad (\text{A.16})$$

Clearly, π is convex and $\pi(m) \geq R(m)$. To see that $\mathbb{E}_G[\pi] = \mathbb{E}_{F_1}[\pi]$, note that

$$\mathbb{E}_{F_1}[\pi(m)] - \mathbb{E}_G[\pi(m)] = \int_{m_*}^{m^*} \pi(m) dm - (m^* - m_*)\pi(\bar{\mu}).$$

Now, because π is linear on $[m_*, m^*]$, we have that

$$\int_{m_*}^{m^*} \pi(m) dm = (m^* - m_*)\pi\left(\frac{m_* + m^*}{2}\right) = (m^* - m_*)\pi(\bar{\mu}).$$

To finish the proof, we show m_*, m^* can be chosen so that $\pi(\bar{\mu}) = R(\bar{\mu})$. The goal is

to find m_*, m^* such that

$$\frac{m_* + m^*}{2} = \bar{\mu} \quad R(\bar{\mu}) = R(m_*) + \frac{R(m^*) - R(m_*)}{m^* - m_*}(\bar{\mu} - m_*). \quad (\text{A.17})$$

Note that the last condition is equivalent to requiring that

$$R(m^*) = R(m_*) + \frac{R(\bar{\mu}) - R(m_*)}{\bar{\mu} - m_*}(m^* - m_*). \quad (\text{A.18})$$

Verifying that m_*, m^* exist that satisfy the above equations is equivalent to showing that the quadratic equation

$$h_0(x) \equiv \frac{2}{c}x^2 - x\left(\frac{2}{c} - 2\bar{\mu} + 1\right) + \frac{1}{2c} + 4\bar{\mu}(1 - \bar{\mu}) - 1 = 0$$

has a solution $m^* \in [1/2, 1]$. It is easy to verify that $h_0(x)$ achieves its minimum at $\frac{1}{2} + (1 - 2\bar{\mu})(c/4) \geq 1/2$. So we need to show that $h_0(2\bar{\mu}) > 0$. This ensures that $m_* = 2\bar{\mu} - m^* > 0$. The latter condition is equivalent to [Equation A.13](#) not holding, which completes the proof.

When [Equation A.13](#) does not hold, $U'(\theta)$ is not monotone, so the disclosure policy that solves the relaxed problem cannot be implemented. \square

Proof of Proposition 3. We now turn to the case $\bar{\mu} \geq 1/2$ and continue to normalize v_H to 1 to simplify notation. Note that regardless of whether $\bar{\mu}$ is below or above $l(c)$, the implied distribution over posteriors G in the policy described in [Proposition 3](#) is as in [Equation A.15](#). Furthermore, conditional on m_*, m^* satisfying [Equation A.17](#), the convex function π defined in [Equation A.16](#) satisfies the conditions in [Dworczak and Martini \(2019\)](#). To finish the proof, we show m_*, m^* can be chosen so that [Equation A.17](#) is satisfied. Consider first the case in which $m_* \leq 0.5$. Verifying that m_*, m^* exist that solve [Equation A.17](#) reduces to showing that the following quadratic equation

$$h_1(x) \equiv \frac{2}{c}x^2 + (2\bar{\mu} - 1 - \frac{2}{c})x - \left[\frac{4\bar{\mu}^2 - 4\bar{\mu}}{c} + \frac{1}{2c} + 2\bar{\mu} - 1 \right] = 0,$$

has a solution $m^* \in (\bar{\mu}, 1]$. It is easy to verify that $h_1(x)$ achieves its minimum at $\frac{1}{2} - \frac{c}{4}(2\bar{\mu} - 1) < \frac{1}{2} < \bar{\mu}$. So we need to show (i) $h_1(\bar{\mu}) < 0$, (ii) $h_1(1) \geq 0$ and (iii) $h_1(2\bar{\mu} - 1/2) < 0$. The last condition ensures $m_* = 2\bar{\mu} - m^* \leq 1/2$.²¹ Tedious but straightforward algebra verifies that a solution exists when $\bar{\mu} \in [1/2, l(c)]$.²²

Consider now the case in which $m_* > 0.5$. In that case, verifying that m_* , m^* exist that solve Equation A.17 reduces to showing that the following quadratic equation

$$h_2(x) \equiv \frac{4}{c}x^2 - x \left[\frac{8}{c}\bar{\mu} + 2\bar{\mu} - 1 \right] + \frac{1}{c} + (2\bar{\mu} - 1)^2 \left(\frac{1}{c} + 1 \right) + \frac{2(2\bar{\mu} - 1)}{c} = 0,$$

has a solution $m_* \in [\frac{1}{2}, \bar{\mu}]$. It is easy to verify $h_2(x)$ achieves its minimum at $\bar{\mu} + (2\bar{\mu} - 1)c/8 > \bar{\mu}$. Hence, we need to show (i) $h_2(\bar{\mu}) < 0$, (ii) $h_2(0.5) \geq 0$, and (iii) $h_2(2\bar{\mu} - 1) \geq 0$. The last condition ensures $m^* = 2\bar{\mu} - m_* \leq 1$.²³ Tedious but straightforward algebra verifies such a solution exists when $\bar{\mu} \in (l(c), 1]$.

We now construct the period 1 allocation and transfers. We abuse notation slightly and write $q_1(m)$ instead of $q_1(F_2)$ when $m = \mu_{F_2}$. When $\bar{\mu} \in (0.5, l(c))$, we have that

$$q_1(m) = \begin{cases} 0 & \text{if } m \leq m_* \\ \frac{2\bar{\mu}-1}{c} & \text{if } m_* \leq m \leq m^* \\ \frac{2m-1}{c} & \text{otherwise} \end{cases} .$$

From Equation A.7 we recover the transfers:

$$x_1(m) = \begin{cases} 0 & \text{if } m \leq m_* \\ m_* \frac{2\bar{\mu}-1}{c} & \text{if } m_* \leq m \leq m^* \\ m \left[\frac{m-m^*}{c} \right] + m_* \frac{2\bar{\mu}-1}{c} - \Delta v m^* + m^* \left[\frac{m+m^*-2\bar{\mu}}{c} \right] & \text{otherwise} \end{cases} .$$

²¹In the knife edge case when $\bar{\mu} = 1/2$, we actually have that $h_1(1/2) = 0$ and $m_* = m^* = 1/2$.

²²Indeed, the expression for $l(c)$ comes from verifying (iii).

²³Note that because (i) implies $\bar{\mu} > m_*$, it automatically follows that $m^* > \bar{\mu}$.

The solution to the relaxed problem induces the following marginal utility for type θ :

$$U'(\theta) = \begin{cases} \Delta v & \text{if } \theta \leq m_* \\ \frac{2\bar{\mu}-1}{c} + \Delta v & \text{if } m_* \leq \theta \leq m^* \\ \frac{2\theta-1}{c} & \text{otherwise} \end{cases} .$$

This satisfies the monotonicity condition only if:

$$m^* \geq \Delta v \frac{c}{2} + \bar{\mu} = (1 - \bar{\mu}) \frac{c}{2} + \bar{\mu}. \quad (\text{A.19})$$

To verify whether Equation A.19 holds, it suffices to check that $h_1(m) \leq 0$ at $m = (1 - \bar{\mu})(c/2) + \bar{\mu}$. This is equivalent to requiring that $\bar{\mu}(1 - \bar{\mu}) \leq 1/(4 + c^2)$. However, this condition is incompatible with $1/2 \leq \bar{\mu} \leq l(c)$. Hence, monotonicity does not hold when $1/2 \leq \bar{\mu} \leq l(c)$. Instead, when $\bar{\mu} \in (l(c), 1)$, we have that

$$q_1(m) = \begin{cases} 0 & \text{if } m \leq \frac{1}{2} \\ \frac{2m-1}{c} & \text{if } m \in (0.5, m_*) \\ \frac{2\bar{\mu}-1}{c} & \text{if } m_* \leq m \leq m^* \\ \frac{2m-1}{c} & \text{otherwise} \end{cases} .$$

From Equation A.7 we recover the transfers:

$$x_1(m) = \begin{cases} 0 & \text{if } m \leq \frac{1}{2} \\ \frac{2m-1}{c} \frac{2m+1}{4} & \text{if } m \in (\frac{1}{2}, m_*) \\ m_* \frac{2\bar{\mu}-1}{c} - (m_* - \frac{1}{2}) \left(\frac{2m_*-1}{c} \right) & \text{if } m_* \leq m \leq m^* \\ m^* \left[\frac{m+m^*-2\bar{\mu}}{c} - \Delta v \right] + m \left(\frac{m-m^*}{c} \right) + m_* \frac{2\bar{\mu}-1}{c} - (m_* - \frac{1}{2}) \frac{1}{2} \left(\frac{2m_*-1}{c} \right) & \text{otherwise} \end{cases} .$$

The solution to the relaxed problem induces the following marginal utility for type θ :

$$U'(\theta) = \begin{cases} \Delta v & \text{if } \theta \leq 0.5 \\ \frac{2\theta-1}{c} + \Delta v & \text{if } 0.5 < \theta < m_* \\ \frac{2\bar{\mu}-1}{c} + \Delta v & \text{if } m_* \leq \theta \leq m^* \\ \frac{2\theta-1}{c} & \text{otherwise} \end{cases} .$$

This satisfies the monotonicity condition if, and only if Equation A.19 holds. Since h_2 is expressed in terms of m_* , Equation A.19 holds only if $m_* \leq \bar{\mu} - \frac{\varepsilon}{2}(1 - \bar{\mu})$. We now find conditions under which (i) $\bar{\mu} - \frac{\varepsilon}{2}(1 - \bar{\mu}) \geq 0.5$ and (ii) $h_2(\bar{\mu} - \frac{\varepsilon}{2}(1 - \bar{\mu})) \leq 0$. For (i) to hold, we need that $\bar{\mu} \geq (1 + c)/(2 + c)$, which is implied by $\bar{\mu} \geq l(c)$. For (ii) to hold we need that $h_2(\bar{\mu} - \frac{\varepsilon}{2}(1 - \bar{\mu})) = (1 - \bar{\mu})(1 + \frac{\varepsilon}{2} - 2\bar{\mu}) \leq 0$, which is the condition in Corollary 2. \square

Figure 9 illustrates the parameter values for which the monotonicity condition holds under the assumption that $v_H = 1$. In the figure, the curve $\bar{\mu} = g(c)$ corresponds to Equation A.13 at equality.

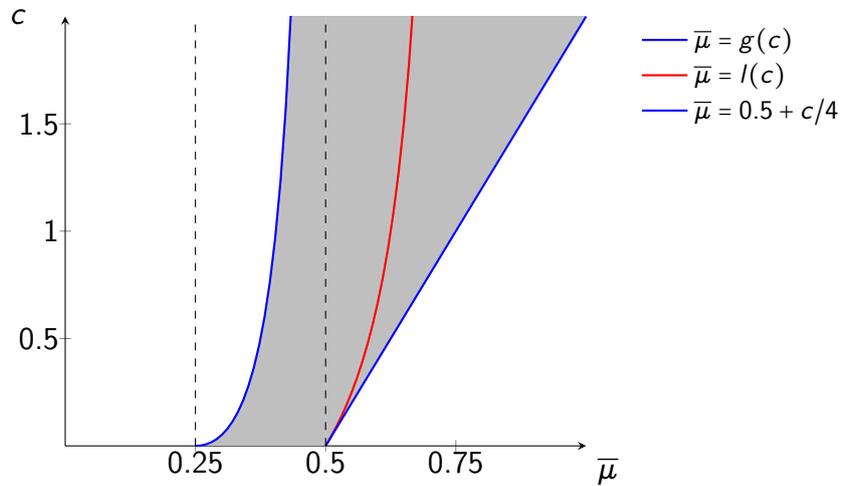


Figure 9: Shaded gray area shows parameter values for which the solution to the relaxed problem does not satisfy monotonicity