Monotone Additive Statistics

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Talk Overview

- Definition of **monotone additive statistics**.
- Characterization.
- Applications.
  - Posted prices for sacks of potatoes.
  - Fishburn-Rubinstein time preferences.
  - Rabin-Weizsäcker preferences over gambles.
- Monotone additive costs of **Blackwell experiments**
  - Different paper: “From Blackwell Dominance in Large Samples to Rényi Divergences and Back Again.”
  - Same authors.
  - Related ideas.
- Work in progress.
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Monotone Additive Statistics

- **A statistic** is a way of capturing distributions by a single number.
  - Expectation.
  - Median.
  - Value at risk.
  - Certainty equivalent.

- Let $L^\infty$ be the set of all bounded random variables.
  - A statistic is a map $\Phi : L^\infty \to \mathbb{R}$ such that
    - $\Phi(c) = c$.
    - If $X$ and $Y$ have the same distribution then $\Phi(X) = \Phi(Y)$.

- It is **monotone** if $X \geq_1 Y$ implies $\Phi(X) \geq \Phi(Y)$.

- Equivalently: it is monotone if $X \geq Y$ implies $\Phi(X) \geq \Phi(Y)$.
  - Because $X \geq_1 Y$ iff $\exists \tilde{X} \sim X, \tilde{Y} \sim Y$ s.t. $\tilde{X} \geq \tilde{Y}$ a.s.

- A statistic is **additive** if $\Phi(X + Y) = \Phi(X) + \Phi(Y)$ whenever $X$ and $Y$ are independent.
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A statistic is additive if $\Phi(X + Y) = \Phi(X) + \Phi(Y)$ whenever $X$ and $Y$ are independent.
**Question:** What are the additive monotone statistics?
Examples of Monotone Additive Statistics

- $\mathbb{E}[X]$.
- $\max[X] = \sup\{c \in \mathbb{R} : \mathbb{P}[X \geq c] > 0\}$.
- $\min[X]$.
- For $a \neq 0$,
  \[ S_a(X) = \frac{1}{a} \log \mathbb{E}[e^{aX}] \]
- By continuity
  - $S_0(X) = \mathbb{E}[X]$,
  - $S_\infty(X) = \max[X]$,
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Characterization

- Is there anything beside the $S_a$’s?
- Main result: this is it.
- Well... we can also take weighted averages.

**Theorem**

Let $\Phi$ be a monotone additive statistic. Then there is a probability measure $m$ on $\mathbb{R} \cup \{+\infty, -\infty\}$ such that

$$
\Phi(X) = \int S_a(X) \, dm(a).
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- $\{S_a\}$ are the extreme points of the set of additive monotone statistics.
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Proof ideas

- Take \( X, Y \) that are not ranked under FOSD.
- Is it possible that there is an independent \( R \) such that \( X + R \geq_1 Y + R \)?
- Example: \( X \sim B(1/3), \ Y \sim U([-3/5, 2/5]) \).

- Works for \( \mathbb{P}[R = \pm 1/5] = 1/2 \).
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![Graph showing probability distribution]

- Works for $\mathbb{P}[R = \pm 1/5] = 1/2$.

![Graph showing another probability distribution]
Proof ideas

- Pomatto, Strack, Tamuz (2019): If $\mathbb{E}[X] > \mathbb{E}[Y]$ then $X + R \geq_1 Y + R$ for some independent $R$.

- Under what conditions on $X, Y$ is there a bounded independent r.v. $R$ such that $X + R \geq_1 Y + R$?

- If $S_a(X) < S_a(Y)$ for some $a$ this is impossible, since

$$S_a(X + R) = S_a(X) + S_a(R) < S_a(Y) + S_a(R) = S_a(Y + R).$$

Theorem

For $X, Y \in L^\infty$, if $S_a(X) > S_a(Y)$ for all $a$, then there exists an $R \in L^\infty$ such that $X + R \geq_1 Y + R$.

- Corollary: if $S_a(X) > S_a(Y)$ for all $a$, then $\Phi(X) \geq \Phi(Y)$, because

$$\Phi(X) + \Phi(R) = \Phi(X + R) \geq \Phi(Y + R) = \Phi(Y) + \Phi(R).$$

- Rest of the proof: exercise in analysis.
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- Pomatto, Strack, Tamuz (2019): If $\mathbb{E}[X] > \mathbb{E}[Y]$ then $X + R \geq_1 Y + R$ for some independent $R$.

- Under what conditions on $X, Y$ is there a bounded independent r.v. $R$ such that $X + R \geq_1 Y + R$?

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- Let $F, G$ be the cdfs of $X, Y$, supported on $[-N, N]$.
- We will find an $R$ with pdf $h$ such that $G \ast h \geq F \ast h$.
- Let $h(x) = e^{-x^2/2V}$. Then

\[
[(G - F) \ast h](y) = \int_{-N}^{N} [G(x) - F(x)] \cdot h(y - x) \, dx \\
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- Works because
  - $e^{-y^2/2V} \approx 1$ for $x \in [-N, N]$ and large $V$.
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$$[(G - F) * h](y) = \int_{-N}^{N} [G(x) - F(x)] \cdot h(y - x) \, dx$$

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**Proof ideas**
**Application: Posted Prices for Sacks of Potatoes**

- Consider a buyer who posts her prices for *potatoes*.
- Farmers come and sell her their crops.

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- Price $P : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.
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So $P(X) = P(1) \cdot \Phi(X)$ for some monotone additive statistic $\Phi$. 

Consider a buyer who posts her prices for sacks of potatoes.

Farmers come and sell her their crops.

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A pair \((x,t)\) is a (positive) amount of money \(x\) at (non-negative) time \(t\). The set of such pairs is \(\Omega = \mathbb{R}^{++} \times \mathbb{R}_+\).

Fishburn and Rubinstein consider preferences \(\succ\) over \(\Omega\).

**Axiom**

1. If \(x \succ y\) then \((x,t) \succ (y,t)\).
2. If \(t \prec s\) then \((x,t) \succ (x,s)\).
3. If \((x,t) \succ (y,s)\) then \((x,t+\tau) \succ (y,s+\tau)\).
4. Upper and lower contour sets are closed.

All such preferences come from exponential discounting.

**Theorem (Fishburn and Rubinstein)**
The axioms imply that \(\succ\) is represented by \(f(x,t) = u(x)e^{-rt}\) for some \(r > 0\), and an increasing \(u: \mathbb{R}^{++} \to \mathbb{R}^{++}\).
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\section*{Axiom}

\begin{itemize}
  \item Keep FR’s axioms for deterministic times.
  \item If \(T <_1 S\) then \((x, T) \succ (x, S)\).
  \item If \((x, T) \succ (y, S)\) then \((x, T + R) \succ (y, S + R)\) for all bounded random independent \(R\).
  \item For all \((x, T)\) there is a \(t\) such that \((x, T) \sim (x, t)\).
\end{itemize}

- All such preferences come from exponential discounting of a monotone additive statistic applied to the random time.

\section*{Theorem}

The axioms imply that \(\succ\) is represented by \(f(x, T) = u(x)e^{-r\Phi(T)}\) for some \(r > 0\), an increasing \(u: \mathbb{R}_{++} \to \mathbb{R}_{++}\), and a monotone additive statistic \(\Phi\).
A pair \((x, T)\) is a (positive) amount of money \(x\) at a random (non-negative) time \(T\).

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Application: Fishburn-Rubinstein Time Preferences

- Example: \( f(x, T) = u(x)\mathbb{E}\left[e^{-rT}\right] \).
  - Expectation of the Fishburn-Rubinstein utility.
  - Agents are risk seeking over time.

- Example: \( f(x, T) = \frac{u(x)}{\mathbb{E}[e^{rT}]} \).
  - \( f(x, T) = u(x)e^{-r\Phi(T)} \) for \( \Phi(T) = \frac{1}{r} \log \mathbb{E}[e^{rT}] \).
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Application: Rabin-Weizsäcker Preferences

- Let $L^\infty$ be the set of bounded gambles.
- Consider an expected utility agent with an increasing utility function $u$ for money.
- Write $X \succ Y$ if $\mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$.

**Axiom**

Suppose $X_1, X_2$ are independent, $Y_1, Y_2$ are independent. If $X_1 \succ Y_1$ and $X_2 \succ Y_2$ then $Y_1 + Y_2$ does not stochastically dominate $X_1 + X_2$.

- What does this tell us about $u$?

**Theorem (Rabin-Weizsäcker)**

The axiom implies that either $u(x) = ae^{ax}$ for some $a \neq 0$, or $u(x) = x$ (up to affine transformations).

- So **CARA** agents are the only ones that satisfy the axiom.
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Application: Rabin-Weizsäcker Preferences

- What about general (non-expected utility) preferences?
  - Write $X \succ Y$ if the agent strictly prefers $X$ to $Y$.

Axiom

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Proposition

The axioms imply that $\succ$ is represented by some monotone additive statistic.

Such preferences can be represented by a monotone additive statistic.

$\Phi$ is the average of CARA certainty equivalents $S_a(X) = \frac{1}{a} \log \mathbb{E} \left[ e^{aX} \right]$. 
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$X + \epsilon \succ X$.

For all $X$ there is a $c \in \mathbb{R}$ such that $X \sim c$.

Such preferences can be represented by a monotone additive statistic.

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Binary state of the world $\theta \in \{0, 1\}$.

A Blackwell Experiment is a pair $\mu = (\mu_0, \mu_1)$ of probability measures on some measurable space $\Omega$.

We say that it is bounded if $\log \frac{d\mu_0}{d\mu_1}$ is bounded.

The collection of bounded experiments is $\mathcal{B}$.

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